Prime filtrations of modules and Stanley decompositions

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Ali Soleyman Jahan

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Vorsitzender:	Prof. Dr. Trung Van Tran
Erster Gutachter:	Prof. Dr. Jürgen Herzog
Zweiter Gutachter:	Prof. Dr. Ralf Fröberg

Dedicated to my wife, my sons Sina and Siamand and my parents.

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Contents

Preface

Filtrations are a useful tool in algebra to study some basic property of objects like modules, rings and groups. In this thesis we study a special class of filtrations, the so-called prime filtrations. Let R be a Noetherian ring, and M a finitely generated R-module. A basic fact in commutative algebra [34, Theorem 6.4] says that there exists a finite filtration

$$\mathcal{F}:\ 0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

with cyclic quotients $M_i/M_{i-1} \cong R/P_i$ and $P_i \in \operatorname{Supp}(M)$. We call any such filtration of M a prime filtration. The set of prime ideals $\{P_1, \ldots, P_r\}$ which define the cyclic quotients of $\mathcal{F}$ will be denoted by $\operatorname{Supp}(\mathcal{F})$. It is easy to see that for any prime filtration $\mathcal{F}$ one has $\operatorname{Ass}(M) \subset \operatorname{Supp}(\mathcal{F}) \subset \operatorname{Supp}(M)$. Let $\operatorname{Min}(M)$ denote the set of minimal prime ideals in $\operatorname{Supp}(M)$. Dress [13] calls a prime filtration $\mathcal{F}$ of M *clean* if $\operatorname{Supp}(\mathcal{F}) = \operatorname{Min}(M)$. The R-module M is called *clean* if it admits a clean filtration.

The concept of *pretty clean modules* was introduced by Herzog and Popescu in [29] as a generalization of the definition of clean modules. A prime filtration

$$\mathcal{F}:\ 0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

of M with $M_i/M_{i-1} \cong R/P_i$ is called *pretty clean*, if for all $i < j$ for which $P_i \subseteq P_j$ it follows that $P_i = P_j$. In other words, a proper inclusion $P_i \subset P_j$ is only possible if $i > j$. The module M is called *pretty clean*, if it has a pretty clean filtration. We say that an ideal $I \subset R$ is pretty clean if R/I is pretty clean. It is clear that clean $\Rightarrow$ pretty clean.

A prime filtration which is pretty clean has the nice property that $\operatorname{Supp}(\mathcal{F}) = \operatorname{Ass}(M)$, see [29, Corollary 3.6]. It is still an open problem to characterize the modules which have a prime filtration $\mathcal{F}$ with $\operatorname{Supp}(\mathcal{F}) = \operatorname{Ass}(M)$.

The concept of clean and pretty clean are the algebraic counterpart of shellability. Let Δ be a simplicial complex on vertex set $[n] = \{1, \ldots, n\}$, and $I_\Delta \subset S = K[x_1, \ldots, x_n]$ the Stanley–Reisner ideal of Δ. Dress [13] showed that Δ is non-pure shellable in the sense of Björner and Wachs [9] if and only if S/I_Δ is clean.

In [29] the authors attach to each monomial ideal a multicomplex and introduce the concept of shellable multicomplexes. In case I is a squarefree monomial ideal, this concept of shellability coincides with non-pure shellability introduced by Björner and Wachs [9]. Herzog and Popescu proved that a multicomplex $\Gamma \subset \mathbb{N}^n_\infty$ is shellable if and only if $S/I(\Gamma)$ is pretty clean, see [29, Theorem 10.5].

This thesis is organized as follows: In the first chapter we recall some definitions, notation and give a short survey of those facts which are relevant in the following chapters. In Chapter 2 we study clean and pretty clean K-algebras. We show that if $I \subset S = K[x_1, \dots, x_n]$ is a monomial ideal of height $\geq n-1$, then S/I is pretty clean. We also notice that if I is a monomial ideal of height 1, then one can write $I = uJ$ for some monomial $u \in S$ and some monomial ideal J with height $J \geq 2$. Then we show that S/I is pretty clean if and only if S/J is pretty clean. By using these facts we show that if I is a monomial ideal in at most three variables, then S/I is pretty clean, see Theorem 2.1.7.

Let $I \subset S$ be a monomial ideal and $I^p \subset T$ its polarization. We denote by Γ and Γ^p the multicomplexes associated to I and I^p. We establish a bijection between the set of facets of Γ and the set of facets of Γ^p. By using this fact we show that S/I is pretty clean if and only if T/I^p is clean. Using this result about polarization we prove that if I is a complete intersection, Cohen–Macaulay of codimension 2 or Gorenstein of codimension 3 monomial ideal, then S/I is clean.

We also give a large and combinatorially interesting class $\mathcal{I}$ of monomial ideals which are pretty clean (Theorem 2.5.5). The class $\mathcal{I}$ is a non-squarefree version of the class of facet ideals of forests in the sense of Faridi [16]. The ideals in $\mathcal{I}$ are called a monomial ideal of forest type. As another consequence of Theorem 2.5.5 we get the main result of [18], which says that $S/I(\Delta)$ is sequentially Cohen-Macaulay for any forest Δ.

We show in Theorem 2.5.12 that I is a monomial ideal of forest type if and only if I has the free variable property. Identifying a squarefree monomial ideal with a clutter, Theorem 2.5.12 says that a clutter has the free vertex property in the sense of Tuyl and Villarreal if and only if the clutter corresponds to a forest in the sense of Faridi, equivalently, a totally balanced clutter in the language of hypergraphs. Let $\mathcal{C}$ be a clutter, and let $\Delta_{\mathcal{C}}$ be the simplicial complex whose Stanley–Reisner ideal is the edge ideal of $\mathcal{C}$. In [53, Theorem 5.3] Villarreal and Tuyl show that $\Delta_{\mathcal{C}}$ is shellable if $\mathcal{C}$ has the free vertex property. Therefore Theorem 2.5.5 may be viewed as a generalization of [53, Theorem 5.3].

In the last part of this chapter we give a lower bound for the length of a prime filtration of S/I. The main result of this chapter is Theorem 2.6.3 which shows that for a monomial ideal $I \subset S$ the following conditions are equivalent:

(a) I is pretty clean;

(b) I^p, the polarization of I is clean;

(c) There exists a prime filtration $\mathcal{F}$ of I with $\ell(\mathcal{F}) = \operatorname{adeg}(I)$;

(d) Γ, the multicomplex associated to I is shellable;

(e) If $\triangle$ be the simplicial complex associated to I^p, then $\triangle$ is shellable.

In Chapter 3 we discuss the Stanley conjecture concerning Stanley decompositions. In [29, Theorem 6.5] it was shown that the Stanley conjecture holds for S/I if S/I is pretty clean. Therefore our results in Section 2 also imply that Stanley's conjecture holds for S/I in the following cases: height $I \geq n-1$, $I \subset K[x_1, x_2, x_3]$,

I is a complete intersection monomial ideal, I is a Cohen–Macaulay monomial ideal of codimension 2, I is a Gorenstein monomial ideal of codimension 3 or I is a monomial ideal of forest type. We notice (Proposition 3.1.4) that for a monomial ideal, instead of requiring that S/I is pretty clean, it suffice to require that there exists a prime filtration $\mathcal{F}$ with $\text{Ass}(S/I) = \text{Supp}(\mathcal{F})$ in order to conclude that the Stanley conjecture holds for S/I.

Unfortunately it is not true that each Stanley decomposition corresponds to a prime filtration as shown by an example of Maclagan and Smith [36, Example 3.8]. However we characterize in Proposition 3.1.9 those Stanley decomposition of S/I that correspond to prime filtrations. Using this characterization we show in Corollary 3.1.12 that in the polynomial ring in two variables Stanley decompositions and prime filtrations are in bijective correspondence.

In Section 3.2 we introduce squarefree Stanley spaces and show in Proposition 3.2.2 that for a squarefree monomial ideal I, the Stanley decompositions of S/I into squarefree Stanley spaces correspond bijectively to partitions into intervals of the simplicial complex whose Stanley-Reisner ideal is the ideal I. Stanley calls a simplicial complex Δ *partitionable* if there exists a partition $\Delta = \bigcup_{i=1}^{r} [F_i, G_i]$ of Δ such that for all intervals $[F_i, G_i] = \{F \in \Delta\colon ; F_i \subset F \subset G_i\}$ one has that G_i is a facet of Δ. Then he conjectured that any Cohen–Macaulay simplicial complex is partitionable, see [47]. We show in Corollary 3.2.5 that if Δ is a Cohen–Macaulay simplicial complex, then Stanley's conjecture on Stanley decompositions is true for S/I_Δ if and only if Δ is partitionable. In other words, Stanley's conjecture on Stanley decompositions implies his conjecture on partitionable simplicial complexes.

Yanagawa [56] introduced squarefree S-modules which generalizes the concept of Stanley–Reisner rings. A finitely generated $\mathbb{N}^n$-graded S-module $M = \bigoplus_{\mathbf{a}\in\mathbb{N}^n} M_{\mathbf{a}}$ is *squarefree* if the multiplication map $M_{\mathbf{a}} \to M_{\mathbf{a}+\varepsilon_i}$, $m \mapsto mx_i$, is bijective for all $\mathbf{a} \in \mathbb{N}^n$ and all $i \in \text{supp}(\mathbf{a})$. Römer defined in [39] the Alexander dual $M^\vee$ for a squarefree S-module M. The definition refers to exterior algebras. Let E be the exterior algebra over an n-dimensional K-vector space V. A finitely generated $\mathbb{N}^n$-graded E-module $N = \bigoplus_{\mathbf{a}\in\mathbb{N}^n} N_{\mathbf{a}}$ is called *squarefree* if it has only squarefree components. By [39, Corollary 1.6] the category of squarefree S-modules is equivalent to the category of squarefree E-modules. For an $\mathbb{N}^n$-graded E-module N the E-dual of N is the graded dual $N^\vee = \text{Hom}_E(N, E)$. Let M be a squarefree S-module and N its corresponding squarefree E-module. Then $M^\vee$ is defined to be the squarefree S-module corresponding to $N^\vee$. Miller [35] defined the Alexander dual even for finitely generated $\mathbb{N}^n$-graded modules. In the case of squarefree modules, Römer's definition coincides with the definition of Miller. We follow the approach of Römer.

In Section 4.1 we study prime filtrations of squarefree S-modules and E-modules. As a main result of this section we prove that for a squarefree S-module M there exists a chain $0 \subset M_1 \subset \cdots \subset M_r = M$ of squarefree submodules of M with $M_i/M_{i-1} \cong S/P_{F_i}(-G_i)$ if and only if there exists a chain $0 \subset L_1 \subset \cdots \subset L_r = M^\vee$ of squarefree submodules of $M^\vee$ with $L_i/L_{i-1} \cong S/P_{G_i}(-F_i)$, see Theorem 4.1.3. For proving this, in Proposition 4.1.2 we show that the corresponding result is true for squarefree E-modules. In Corollary 4.1.4 we show explicitly how the prime filtration of $M^\vee$ is obtained from that of M, in the special case that $M = J/I$, where $I \subset J$ are squarefree monomial ideals.

In Section 4.2 we study Stanley decompositions of finitely generated $\mathbb{Z}^n$-graded S-modules. As a main result we show that a squarefree S-module M has a squarefree Stanley decomposition $M = \bigoplus_{i=1}^t m_iK[Z_i]$ if and only if there exists a squarefree Stanley decomposition $M^\vee = \bigoplus_{i=1}^t v_iK[W_i]$ of $M^\vee$ with $\mathrm{supp}(v_i) = [n] \setminus \{j \colon x_j \in Z_i\}$ and $W_i = \{x_j \colon j \in [n] \setminus \mathrm{supp}(m_i)\}$, see Theorem 4.2.6. To prove this we show in Proposition 4.2.3 that the corresponding result is true for squarefree E-modules. As corollaries of Theorem 4.2.6 we show that Stanley's conjecture on Stanley decompositions holds for a squarefree S-module M if and only if $M^\vee$ has a Stanley decomposition $M^\vee = \bigoplus_{i=1}^t v_iK[W_i]$ with $|v_i| \leq \mathrm{reg}(M^\vee)$ for all i, see Corollary 4.2.7. On the other hand Stanley's conjecture on partitionable simplicial complexes holds for a Cohen–Macaulay simplicial complex Δ if and only if $I_{\Delta^\vee}$ has a Stanley decomposition $I_{\Delta^\vee} = \bigoplus_{i=1}^t u_iK[Z_i]$ such that $\{u_i, \ldots, u_t\} = G(I_{\Delta^\vee})$.

Due to these facts we conjecture (Conjecture 4.2.9) that any $\mathbb{Z}^n$-graded S-module M has a Stanley decomposition $M = \bigoplus_{i=1}^t m_iK[Z_i]$ with $|m_i| \leq \mathrm{reg}(M)$. In some cases we can show that this conjecture holds.

Let $I \subset S$ be a monomial ideal. We denote by $G(I)$ the unique minimal monomial system of generators of I. We say that I has linear quotients, if there exists an order $\sigma = u_1, \ldots, u_m$ of $G(I)$ such that the ideal $(u_1, \ldots, u_{i-1}) : u_i$ is generated by a subset of the variables for $i = 2, \ldots, m$. We denote this subset by $q_{u_i,\sigma}(I)$. Any order of the generators for which we have linear quotients will be called an admissible order. Ideals with linear quotients were introduced by Herzog and Takayama [33]. If each component of I has linear quotients, then we say I has componentwise linear quotients.

The concept of linear quotients, similarly as the concept of shellability, is purely combinatorial. However both concepts have strong algebraic implications. Indeed, an ideal with linear quotients has a componentwise linear resolution while shellability of a simplicial complex implies that it is sequentially Cohen-Macaulay. These similarities are not accidental. In fact, let Δ be a simplicial complex and I_Δ its Stanley-Reisner ideal. It is well-known that I_Δ has linear quotients if and only if the Alexander dual of Δ is shellable. Thus at least in the squarefree case "linear quotients" and "shellability" are dual concepts. On the other hand, linear quotients are not only defined for squarefree monomial ideals, and hence this concept is more general than that of shellability.

In the last section we prove some fundamental properties of monomial ideals with linear quotients. In general, the product of two ideals with linear quotients need not to have linear quotients, even if one of them is generated by a subset of the variables, see Example 5.1.4. However in Lemma 5.1.5, we show that if $I \subset S$ is a monomial ideal with linear quotients, then $\mathfrak{m}I$ has linear quotients, where $\mathfrak{m} = (x_1, \ldots, x_n)$ is the maximal graded ideal of S.

Let I be a monomial ideal with linear quotients and $\sigma = u_1, \ldots, u_m$ an admissible order of $G(I)$. It is not hard to see that $\deg u_i \geq \min\{\deg u_1, \ldots, \deg u_{i-1}\}$, for all $i \in [m] = \{1, \ldots, m\}$. But this order need not to be a degree increasing order. We show in Lemma 5.1.1, that there exists a degree increasing admissible order σ' induced by σ. Furthermore, one has $q_{u,\sigma}(I) = q_{u,\sigma'}(I)$ for any $u \in G(I)$, see Proposition 5.1.2. This implies in particular the "Rearrangement Lemma" of Björner and Wachs [9].

As a main result of Section 5, we show in Theorem 5.1.7, that any monomial ideal with linear quotients has componentwise linear quotients, and hence it is componentwise linear. Conversely, assuming that all components of I have linear quotients, we can prove that I has linear quotients only under some extra assumption, see Proposition 5.1.9. It would be of interest to know whether the converse of Theorem 5.1.7 is true in general.

Herzog and Hibi showed in [21] that a squarefree monomial ideal I is componentwise linear if and only if the squarefree part of each component has a linear resolution. We would like to remark that the "only if" part of this statement is true more generally. Indeed for *any* componentwise linear monomial ideal, the squarefree part of each component has a linear resolution. Here we prove a slightly different result by showing that if a monomial ideal I has linear quotients, then the squarefree part of I has linear quotients. This together with Theorem 5.1.7 implies that the squarefree part of each component of I has again linear quotients. As a corollary of the above facts we obtain that if Δ is shellable, then each facet skeleton of Δ is shellable. Unless Δ is pure, this result differs from the well-known fact that each skeleton of a shellable simplicial complex is again shellable.

1 Preliminaries

In this chapter we collect some basic facts which will be used throughout of this thesis. Throughout this work all rings are assumed to be commutative, Noetherian and all modules are finitely generated unless otherwise stated.

1.1 Graded rings and graded modules

In this section we recall some definitions and some basic facts concerning graded rings and modules.

Definition 1.1.1. Let $(G,+)$ be an abelian group. A ring R is called G-graded if there exists a family of $\mathbb{Z}$-modules R_g, $g \in G$ such that $R = \bigoplus_{g\in G} R_g$ as a $\mathbb{Z}$-module with $R_gR_h \subseteq R_{g+h}$ for all $g, h \in G$. Let R be a graded ring. An R-module M is called G-graded if there exists a family of $\mathbb{Z}$-modules M_g, $g \in G$ such that $M = \bigoplus_{g\in G} M_g$ as a $\mathbb{Z}$-module with $R_gM_h \subseteq M_{g+h}$ for all $g, h \in G$.

We call $u \in M$ homogeneous of degree g if $u \in M_g$ for some $g \in G$ and set $\deg(u) = g$. For $g \in G$ we say that M_g is a homogeneous component of M of degree g. An ideal $I \subset R$ is G-graded if $I = \bigoplus_{g\in G} I_g$ with $I_g = I \cap R_g$.

Definition 1.1.2. Let R be a G-graded ring and M, N are G-graded R-modules. An R-linear map $\varphi : M \to N$ is said to be graded (or homogeneous) of degree h for some $h \in G$ if $\varphi(M_g) \subseteq N_{g+h}$ for all $g \in G$. We call φ homogeneous if it is homogeneous of degree 0.

Definition 1.1.3. Let R be a G-graded ring, M be a G-graded R-module and $g \in G$. We define $M(g)$ to be the G-graded R-module M with

$$M(g)_h = M_{g+h} \quad \text{for all} \quad h \in G.$$

We call $M(g)$ the g-th twist of M. Note that, if $\varphi : M \to N$ is homogeneous of degree h, then the induced map $\widetilde{\varphi} : M(-h) \to N$ is homogeneous.

If G equals $\mathbb{Z}$ or $\mathbb{Z}^n$, we say that R is a graded or a $\mathbb{Z}^n$-graded ring and M is a graded or a $\mathbb{Z}^n$-graded R-module.

Example 1.1.4.

(i) Let $S = K[x_1, \dots, x_n]$ be a polynomial ring over a field K and $I \subset S$ be a monomial ideal. If we set $\deg(x_i) = 1$, then S is a graded ring and $I, S/I$ are graded S-modules. If we set $\deg(x_i) = \varepsilon_i$ where ε_i denotes the i-th unit vector of $\mathbb{Z}^n$, then S is a $\mathbb{Z}^n$-graded ring and I and S/I are $\mathbb{Z}^n$-graded S-modules.

(ii) Let $E = K\langle e_1, \dots, e_n\rangle$ be the exterior algebra over an n-dimensional K-vector space V with basis $e_1, \cdots, e_n$. Then E has a graded structure induced by $\deg(e_i) = 1$. We say that $J \subset E$ is a monomial ideal if $J = \langle e_{i_1} \cdots e_{i_r} : i_1 < i_2 < \ldots < i_r\rangle$. If $J \subset E$ is a monomial ideal, then J and E/J are graded E-modules. Note that E has also a $\mathbb{Z}^n$-graded structure induced by $\deg(e_i) = \varepsilon_i$. In this situation, J and E/J are $\mathbb{Z}^n$-graded E-modules.

A graded ring R is called *standard graded* if R is generated by elements of degree 1 over R_0, that is, $R = R_0[R_1]$. From now on all graded rings are assumed to be standard graded unless otherwise stated.

Let R be a graded ring and M be a finitely generated graded R-module. The numerical function

$$H(M,-)\colon \mathbb{Z} \to \mathbb{Z} \quad \text{with} \quad H(M,i) = \ell(M_i) \quad \text{for all} \quad i \in \mathbb{Z},$$

where $\ell(M_i)$ denotes the length of M_i as an R_0-module, is called the Hilbert function of M and

$$\mathrm{Hilb}(M) = \sum_{i\in\mathbb{Z}} \ell(M_i)t^i$$

is called the Hilbert series of M.

Example 1.1.5. Let $S = K[x_1, \dots, x_n]$ be a the polynomial ring over a field K and $\deg x_i = 1$ for $i = 1, \dots, n$. Then one has

$$H_R(i) = \dim_K R_i = \binom{i+n-1}{n-1},$$

and for $n \geq 1$,

$$\mathrm{Hilb}(R) = \sum_{i=0}^{\infty} \dim_K R_i t^i = \sum_{i=0}^{\infty} \binom{i+n-1}{n-1} t^i = \frac{1}{(1-t)^n}.$$

Theorem 1.1.6. *(Hilbert). Let R be a graded ring and M be a finitely generated graded R-module of dimension d. Then $H(M,i)$ is polynomial of degree $d-1$ for large i.*

It is well known that for any non-zero finitely generated graded module M over a standard graded ring there exists a unique $Q_M(t) \in \mathbb{Z}[t, t^{-1}]$ with $Q_M(1) \neq 0$ such that the Hilbert series of M is always a rational function of the form

$$\mathrm{Hilb}(M) = \frac{Q_M(t)}{(1-t)^d},$$

where d denotes the Krull-dimension of M, see [7, Corollary 4.1.8]. The number $Q_M(1)$ is called the *multiplicity* of M and denoted by $e(M)$.

Associated to any graded R-module M there exists a unique (up to isomorphism) minimal graded free resolution

$$0 \to \bigoplus_j R(-j)^{\beta_{r,j}(M)} \to \cdots \to \bigoplus_j R(-j)^{\beta_{1,j}(M)} \to \bigoplus_j R(-j)^{\beta_{0,j}(M)} \to M \to 0.$$

The number $\beta_{i,j}(M)$ is called the graded Betti number of M. The regularity of M is

$$\mathrm{reg}(M) = \max\{j - i \colon \text{for all } i, j\}.$$

If R is $\mathbb{Z}^n$-graded and M a finitely generated $\mathbb{Z}^n$-graded R-module, then we can associate to M its *minimal $\mathbb{Z}^n$-graded free resolution*

$$0 \to \bigoplus_j R(-\mathbf{a}_j)^{\beta_{r,j}(M)} \to \cdots \to \bigoplus_j R(-\mathbf{a}_j)^{\beta_{1,j}(M)} \to \bigoplus_j R(-\mathbf{a}_j)^{\beta_{0,j}(M)} \to 0.$$

The number $\beta_{i,j}(M)$ is the *ij-th graded Betti number* of M. The regularity of M is

$$\operatorname{reg}(M) = \max\{|\mathbf{a}_j| - i \colon \text{for all } i, j\},$$

where $|\mathbf{a}| = \sum_{i=1}^n a_i$ for $\mathbf{a} = (a_1, \ldots, a_n)$.

1.2 Dimension filtration and Cohen–Macaulay filtration

Let R be a Noetherian ring, and M a finitely generated R-module of dimension d. Peter Schenzel in [41] introduced the dimension filtration.

Definition 1.2.1. A filtration

$$\mathcal{F}:\ 0 \subset D_0(M) \subset D_1(M) \subset \cdots \subset D_d(M) = M$$

of M with the property that $D_i(M)$ is the largest submodule of M such that $\dim D_i(M) \leq i$ is called the *dimension filtration* of M.

It is convenient to set $D_{-1}(M) = (0)$. We set

$$\operatorname{Ass}^i(M) = \{P \in \operatorname{Ass}(M) \colon\ \dim R/P = i\}$$

for $i = 1, \ldots, d$. The following characterization of dimension filtration is due to Herzog and Popescu [29, Proposition 1.1].

Proposition 1.2.2. *Let $\mathcal{F}:\ 0 = M_0 \subset M_1 \subset \cdots \subset M_d = M$ be a filtration of M. Then $\mathcal{F}$ is a dimension filtration of M if and only if $\operatorname{Ass}^i(M) = \operatorname{Ass}(M_i/M_{i-1})$.*

As a consequence of Proposition 1.2.2 we can compute $D_i(M)$ from a primary decomposition of (0) in M. More precisely we have

Corollary 1.2.3. *(Schenzel) If $(0) = \bigcap_{i=1}^n Q_i$ is a primary decomposition of (0) in M where Q_i is P_i-primary, then*

$$D_i(M) = \bigcap_{\dim R/P_j \geq i+1} Q_i$$

for $i = 1, \ldots, \dim M$.

Example 1.2.4. Let $S = K[a,b,c,d]$ be the polynomial ring over the field K, and $I \subset S$ the ideal

$$I = (a,b)\cdot(c,d)\cdot(a,c,d) = (abc, abd, acd, ad^2, a^2d, ac^2, a^2c, bcd, bc^2, bd^2).$$

Let $M = S/I$. Then

$$(a,b)\cap(c,d)\cap(a,c,d^2)\cap(a,c^2,d)\cap(a^2,b,c,d^2)\cap(a^2,b,c^2,d)$$

modulo I is an irredundant primary decomposition of (0) in M. Therefore $D_{-1}(M) = 0$, $D_0(M) = ((a,b)\cap(c,d)\cap(a,c,d^2)\cap(a,c^2,d))/I$, $D_1(M) = ((a,b)\cap(c,d))/I$ and $D_2(M) = M$.

Let $(R,\mathfrak{m})$ be a Noetherian local ring, or a standard graded K-algebra with graded maximal ideal $\mathfrak{m}$. Furthermore let M be a finitely generated and graded if R is graded R-module. The following definition is due to Stanley [47, Section II, 3.9] and Schenzel [41].

Definition 1.2.5. Let M be a finitely generated (graded) R-module. A finite filtration

$$\mathcal{F}:\ 0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

of M by (graded) submodules of M is called a *Cohen–Macaulay filtration* if M_i/M_{i-1} is Cohen–Macaulay and

$$\dim M_1/M_0 \le \dim M_2/M_1 \le \ldots \le \dim M_r/M_{r-1}.$$

The R-module M is called *sequentially Cohen–Macaulay* if M admits a Cohen–Macaulay filtration.

The proof of the following fact in the graded case can be found in [30].

Proposition 1.2.6. *Let R be a Cohen–Macaulay of dimension n with the canonical module w_R. Assume that M is sequentially Cohen–Macaulay with a Cohen–Macaulay filtration. Furthermore suppose that $d_i = \dim(M_i/M_{i-1})$ for $i = 1,\ldots,r$. Then*

(a) $\operatorname{Ext}_R^{n-d_i}(M, w_R) \cong \operatorname{Ext}_R^{n-d_i}(M_i/M_{i-1}, w_R)$;

(b) $\operatorname{Ext}_R^{n-d_i}(M, w_R)$ *is Cohen–Macaulay of dimension d_i for $i = 1,\ldots,r$;*

(c) $\operatorname{Ext}_R^{j}(M, w_R) = 0$ *if $j \notin \{n-d_1,\ldots,n-d_r\}$;*

(d) $\operatorname{Ext}_R^{n-d_i}(\operatorname{Ext}_R^{n-d_i}(M, w_R), w_r) \cong M_i/M_{i-1}$ *for $i = 1,\ldots,r$.*

Corollary 1.2.7. *With assumption as in Proposition 1.2.6 one has*

$$\operatorname{Ass}(\operatorname{Ext}_R^{j}(M, w_R)) = \operatorname{Ass}(M_i/M_{i-1}).$$

The following result is due to Schenzel [41, Corollary 2.3].

Theorem 1.2.8. *Let M be sequentially Cohen–Macaulay. Then* $\mathrm{Ass}(M_i/M_{i-1}) = \mathrm{Ass}^{d_i}(M)$ *for all i. In particular* $\mathrm{Ass}(M) = \bigcup_i \mathrm{Ass}(M_i/M_{i-1})$.

As a consequence of Proposition 1.2.2 and Theorem 1.2.8 we have

Corollary 1.2.9. *[41, Proposition 4.3] An R-module M is sequentially Cohen–Macaulay if and only if the factors in the dimension filtration of M are 0 or Cohen–Macaulay.*

1.3 Clean and pretty clean modules

Let R be a Noetherian ring, and M a finitely generated R-module. It is known that there exists a finite filtration

$$\mathcal{F}: \ 0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

with cyclic quotients $M_i/M_{i-1} \cong R/P_i$ and $P_i \in \mathrm{Supp}(M)$, see [34, Theorem 6.4]. We call any such filtration of M a prime filtration. The set of prime ideals $P_1, \ldots, P_r$ which define the cyclic quotients of $\mathcal{F}$ will be denoted by $\mathrm{Supp}(\mathcal{F})$. We denote by $\ell(\mathcal{F})$ the length of the prime filtration $\mathcal{F}$. It is easy to see that if $\mathcal{F}$ is a prime filtration of M, then $\mathrm{Ass}(M) \subset \mathrm{Supp}(\mathcal{F})$. Indeed from this prime filtration of M we get the prime filtration

$$\mathcal{F}_1: \ 0 = M_1/M_1 \subset M_2/M_1 \subset \cdots \subset M_r/M_1 = M/M_1$$

of M/M_1 with $\ell(\mathcal{F}_1) = r-1$. Then if we use induction on the length of filtration, by induction hypothesis $\mathrm{Ass}(M/M_1) \subset \mathrm{Supp}(\mathcal{F}_1)$. On the other hand from the exact sequence

$$0 \to M_1 \to M \to M/M_1 \to 0$$

and [34, Theorem 6.3] we have

$$\mathrm{Ass}(M) \subset \mathrm{Ass}(M_1) \cup \mathrm{Ass}(M/M_1) \subset \{P_1\} \cup \mathrm{Supp}(\mathcal{F}_1) = \mathrm{Supp}(\mathcal{F}).$$

Let $\mathrm{Min}(M)$ denote the set of minimal prime ideals in $\mathrm{Supp}(M)$. Dress [13] calls a prime filtration $\mathcal{F}$ of M *clean* if $\mathrm{Supp}(\mathcal{F}) = \mathrm{Min}(M)$. The R-module M is called *clean* if it admits a clean filtration. Cleanness is the algebraic counterpart of shellability. In [29, Lemma 3.1], Herzog and Popescu gave the following characterization of clean filtration.

Lemma 1.3.1. *Let $\mathcal{F}$ be a prime filtration of M. Then $\mathcal{F}$ is a clean filtration of M if and only if for all P and Q in* $\mathrm{Supp}(\mathcal{F})$ *if $P \subset Q$, then $P = Q$.*

The concept of *pretty clean modules* was introduced by Herzog and Popescu in [29] by weakening the "only if" part of Lemma 1.3.1 as in the following:

Definition 1.3.2. A prime filtration

$$\mathcal{F}: \ 0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

of M with $M_i/M_{i-1} \cong R/P_i$ is called *pretty clean*, if for all $i < j$ for which $P_i \subseteq P_j$ it follows that $P_i = P_j$. In other words, a proper inclusion $P_i \subset P_j$ is only possible if $i > j$.

The module M is called *pretty clean*, if it has a pretty clean filtration. We say that an ideal $I \subset R$ is pretty clean if R/I is pretty clean.

Lemma 1.3.3. *[29, Lemma 3.3] Let $\mathcal{F}: \ 0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$ be a pretty clean filtration of M. Then $P_i \in \mathrm{Ass}(M_i)$ for $i = 1, \ldots, r$.*

As a consequence of this lemma we get the following nice property of pretty clean filtrations.

Corollary 1.3.4. *If $\mathcal{F}$ is a pretty clean filtration of M, then $\mathrm{Ass}(M) = \mathrm{Supp}(\mathcal{F})$.*

It is easy to see that clean modules are pretty clean and if an R-module M is clean, then $\mathrm{Min}(M) = \mathrm{Ass}(M)$. On the other hand if an R-module M has no embedded associated prime ideal, then M is pretty clean if and only if M is clean. Therefore a pretty clean module M is clean if and only if $\mathrm{Min}(M) = \mathrm{Ass}(M)$.

In [29, Corollary 4.2] the authors showed that the relationship between pretty clean modules and sequentially modules is as in the following:

Proposition 1.3.5. *Let M be a sequentially Cohen–Macaulay R-module. If the non-zero factors of the dimension filtration of M are clean, then M is pretty clean.*

Conversely assume that R is a local or graded Cohen–Macaulay ring with the canonical module w_R, and that M admits a pretty clean filtration $\mathcal{F}$ such that R/P is Cohen–Macaulay for all $P \in \mathrm{Supp}(\mathcal{F})$. Furthermore assume that M is graded if R is graded. Then the non-zero factors in the dimension filtration of M are clean.

Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring and $I \subset S$ a monomial ideal. The following result is due to Herzog and Popescu [29, Corollary 4.3].

Theorem 1.3.6. *The following conditions are equivalent:*

(a) *S/I is pretty clean;*

(b) *S/I is sequentially Cohen–Macaulay, and the non-zero factors in the dimension filtration of S/I are clean;*

(c) *The non-zero factors in the dimension filtration of S/I are clean.*

1.4 Simplicial complexes and Stanley–Reisner ring

In this section we fix the terminology and review some notation on simplicial complexes.

A *simplicial complex* Δ over the set of vertices $[n] = \{1, \ldots, n\}$ is a collection of subsets of $[n]$ with the property that $i \in \Delta$ for all $\{i\} \in [n]$, and if $F \in \Delta$ then all the subsets of F are also in Δ (including the empty set). An element of Δ is called a *face* of Δ, and the maximal faces of Δ under inclusion are called *facets*. We denote by $\mathcal{F}(\Delta)$ the set of facets of Δ. The simplicial complex with facets $F_1, \ldots, F_m$ is denoted by $\langle F_1, \ldots, F_m \rangle$. The *dimension* of a face F is defined as $|F| - 1$, where $|F|$ is the number of vertices of F. The dimension of the simplicial complex Δ is the maximal dimension of its facets. A simplicial complex Δ is called *pure* if all facets of Δ have the same dimension, namely $\dim \Delta$. A simplicial complex Γ is called a *subcomplex* of Δ if $\mathcal{F}(\Gamma) \subset \mathcal{F}(\Delta)$.

Let Δ be a simplicial complex on the vertex set $[n]$ and K a field. We denote by $S = K[x_1, \ldots, x_n]$ the polynomial ring in n variables over a field K. The Stanley–Reisner ideal I_Δ is a squarefree monomial ideal generated by monomials $x_{i_1} \cdots x_{i_t}$ such that $\{i_1, \ldots, i_t\} \notin \Delta$. The Stanley–Reisner ring $K[\Delta] = S/I_\Delta$ is well studied, see for example [47], [7] or [20] for details.

Theorem 1.4.1. *Let Δ be a simplicial complex over the vertex set $[n]$. Then*

$$I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} P_{F^c},$$

where $P_{F^c} = (x_j \colon j \notin F)$. In particular $\dim K[\Delta] = \dim \Delta + 1$.

Let $\dim \Delta = d - 1 \geq 0$. We denote by f_i the number of i-dimensional faces of Δ. We have $f_0 = n$ and $f_{-1} = 1$. The d-tuple

$$f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$$

is called the *f-vector* of Δ.

Recall that a graded K-algebra R of dimension d has a Hilbert series of the form

$$\mathrm{Hilb}(R) = \frac{Q_M(t)}{(1-t)^d}$$

where $Q(t)$ is a polynomial with integer coefficients. In principle

$$\mathrm{Hilb}(K[\Delta]) = \frac{h_0 + h_1 t + \cdots + h_d t^d}{(1-t)^d}.$$

The sequence $h(\Delta) = (h_0, \cdots, h_d)$ is called the h-vector of Δ.

Lemma 1.4.2. *The h-vector and the f-vector of a $(d-1)$-dimensional simplicial complex Δ are related by $\sum_{i=0}^{d} h_i s^i (1+s)^{d-i} = \sum_{i=0}^{d} f_{i-1} s^i$.*

A subset C of $[n]$ is called a *vertex cover* of Δ, if $C \cap F \neq \emptyset$ for all facets F of Δ. A vertex cover C is said to be minimal if no proper subset of C is a vertex cover of Δ. Recently, vertex cover algebras have studied in [24], [25] and [26].

Another squarefree monomial ideal associated to Δ, th so-called facet ideal, was first studied by Faridi [16]. The ideal $I(\Delta)$ generated by all monomials $x_{i_1} \cdots x_{i_s}$, where $\{i_1, \ldots, i_s\}$ is a facet of Δ, is called the *facet ideal of* Δ. For a simplicial complex of dimension 1, the facet ideal is the *edge ideal*, which was first studied by Villarreal [55].

The following definitions were first introduced by Faridi in [16]. Let Δ be a simplicial complex. A facet F of Δ is called a *leaf* if either F is the only facet of Δ, or there exists a facet $G \neq F$ in Δ such that $F \cap H \subseteq F \cap G$ for any facet $H \in \Delta$, $H \neq F$. The facet G is called a *branch* of F. A simplicial complex Δ is called a *tree* if it is connected and every nonempty subcomplex of Δ has a leaf. A simplicial complex Δ with the property that every connected component is a tree is called a *forest*. A vertex $t \in F$ is called a free vertex of F if $F \in \mathcal{F}(\Delta)$ is the unique facet which contains t. It is easy to see that any leaf has a free vertex.

Recall that the *Alexander dual* $\Delta^\vee$ of a simplicial complex Δ is the simplicial complex whose faces are $\{[n] \setminus F\colon\ F \notin \Delta\}$. Let I be a squarefree monomial ideal in S. We denote by $\tilde{I}$ the squarefree monomial ideal which is minimally generated by all monomials $x_{i_1} \cdots x_{i_k}$, where $(x_{i_1}, \ldots, x_{i_k})$ is a minimal prime ideal of I. It is easy to see that for any simplicial complex Δ, one has $I_{\Delta^\vee} = \tilde{I}_\Delta$. Let

$$\Delta^c = \langle [n] \setminus F\colon F \in \mathcal{F}(\Delta) \rangle.$$

It is easy to see that $\tilde{I}_\Delta = I_{\Delta^\vee} = I(\Delta^c)$, see [28].

For any set $F \subset [n]$, we denote by $x_F = \prod_{j \in F} x_j$ the squarefree monomial in S whose support is F. In general, for any monomial $u \in S$, the *support* of u is $\mathrm{supp}(u) = \{j\colon x_j \mid u\}$.

Remark 1.4.3. Let Δ be a simplicial complex on $[n]$. Then

$$G(\tilde{I}(\Delta)) = \{x_U = \prod_{j \in U} x_j\colon \text{ where } U \text{ is a minimal vertex cover of } \Delta\}.$$

Now we recall the definition of (non-pure) shellable simplicial complex. According to [9] an order $F_1, \ldots, F_t$ of the facets of Δ is called a *shelling* if and only if for every $1 \leq i < k \leq t$ there exists a j with $1 \leq j < k$ and an $l \in F_k$ such that

$$F_i \cap F_k \subset F_j \cap F_k = F_k \setminus \{l\}.$$

Next we recall the following important fact which is due to Dress [13].

Theorem 1.4.4. *Let Δ be a simplicial complex and $I = I_\Delta \subset S$ its Stanley-Reisner ideal. Then the simplicial complex Δ is (non-pure) shellable if and only if S/I_Δ is clean.*

The following notion is important for our later discussion. Let $I = (u_1, \ldots, u_m)$ be a monomial ideal in S. According to [32], the monomial ideal I has linear quotients if one can order the set of minimal generators of I, $G(I) = \{u_1, \ldots, u_m\}$, such that the ideal $(u_1, \ldots, u_{i-1}) : u_i$ is generated by a subset of the variables for $i = 2, \ldots, m$. This means for each $j < i$, there exists a $k < i$ such that $u_k : u_i = x_t$ and $x_t \mid u_j : u_i$, where $t \in [n]$ and $u_k : u_i = u_k / \gcd(u_k, u_i)$. In the case that I is squarefree, it is enough to show that for each $j < i$, there exists a $k < i$ such that $u_k : u_i = x_t$ and $x_t \mid u_j$. Such an order of generators is called an *admissible order* of $G(I)$. Let $\sigma = u_1, \ldots, u_m$ be an admissible order of $G(I)$. We denote by $q_{u_j,\sigma}(I) \subset \{x_1, \ldots, x_n\}$ the set of minimal generators of $(u_1, \ldots, u_{j-1}) : u_j$.

It is known that if I is a monomial ideal with linear quotients and generated in one degree, then I has a linear resolution, see [57].

Remark 1.4.5. For an ideal which has linear quotients, there might exist several admissible orders. For example, let $I = (x_1x_2, x_1x_3^2x_4, x_2x_4) \subset K[x_1, x_2, x_3, x_4]$. Then $\sigma_1 = x_1x_2, x_1x_3^2x_4, x_2x_4$ and $\sigma_2 = x_1x_2, x_2x_4, x_1x_3^2x_4$ both are admissible orders of $G(I)$.

The following result relates squarefree monomial ideals with linear quotients to (non-pure) shellable simplicial complexes.

Theorem 1.4.6. *[28, Theorem 1.4] Let Δ be a simplicial complex and $I_{(\Delta)^\vee}$ the Stanley-Reisner ideal of $\Delta^\vee$. Then Δ is (non-pure) shellable if and only if $I_{(\Delta)^\vee}$ has linear quotients.*

Combining Theorem 1.4.4 and Theorem 1.4.6, we get the following

Corollary 1.4.7. *Let $I \subset S$ be a squarefree monomial ideal. Then S/I is clean if and only if $\tilde{I}$ has linear quotients.*

1.5 Multicomplexes

The aim of the section is to recall the definition and some basic facts about multicomplexes. In this thesis by $\mathbb{N}$ we always mean $\mathbb{Z}_{\geq 0}$. Stanley [47] called a subset $\Gamma \subset \mathbb{N}^n$ a multicomplex if for all $\mathbf{a} \in \mathbb{N}^n$ and for all $\mathbf{b} \leq \mathbf{a}$, i.e $\mathbf{a}(i) \leq \mathbf{b}(i)$ for $i = 1, \ldots, n$, one has $\mathbf{b} \in \mathbb{N}^n$. Herzog and Popescu [29] gave the following modification of Stanley's definition of multicomplex which will be used in this thesis. Before we give this definition we introduce some notation. We set $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$. Let Γ be a subset of $\mathbb{N}_\infty^n$. The elements of Γ are called *faces*. An element $\mathbf{m} \in \Gamma$ is called maximal if there is no $\mathbf{a} \in \Gamma$ with $\mathbf{a} > \mathbf{m}$. We denote by $M(\Gamma)$ the set of maximal elements of Γ. If $\mathbf{a} \in \Gamma$, we call

$$\operatorname{infpt}(\mathbf{a}) = \{i : \ a(i) = \infty\},$$

the *infinite part* of $\mathbf{a}$. The following simple fact is very important.

Lemma 1.5.1. *[29, Lemma 9.1] If $\Gamma \subset \mathbb{N}^n_\infty$, then $M(\Gamma)$ is a finite set.*

Definition 1.5.2. A subset $\Gamma \subset \mathbb{N}^n_\infty$ is called a *multicomplex* if

(i) for all $\mathbf{a} \in \Gamma$ and for all $\mathbf{b} \leq \mathbf{a}$ it follows that $\mathbf{b} \in \Gamma$,

(ii) for all $\mathbf{a} \in \Gamma$ there exists an element $\mathbf{m} \in M(\Gamma)$ such that $\mathbf{a} \leq \mathbf{m}$.

Note that $\Delta(\Gamma) = \{\operatorname{infpt} \mathbf{a} \colon \mathbf{a} \in \Gamma\}$ is a simplicial complex on vertex set $[n]$ and it is called the simplicial complex associated to Γ.

The number $\dim \mathbf{a} = |\operatorname{infpt} \mathbf{a}|$ is called the *dimension* of $\mathbf{a}$. The dimension of Γ is defined to be $\dim \Gamma = \max\{\dim \mathbf{a} \colon \mathbf{a} \in \Gamma\}$. Obviously one has $\dim \Gamma = \dim \Delta(\Gamma)$.

An element $\mathbf{a} \in \Gamma$ is called a *facet* of Γ if for all $\mathbf{m} \in M(\Gamma)$ with $\mathbf{a} \leq \mathbf{m}$, one has $\operatorname{infpt}(\mathbf{a}) = \operatorname{infpt}(\mathbf{m})$. The set of all facets of Γ will be denoted by $F(\Gamma)$. The facets in $M(\Gamma)$ are called *maximal facets*. It is clear that $M(\Gamma) \subset F(\Gamma)$.

Lemma 1.5.3. *[29, Lemma 9.6] Each multicomplex has a finite number of facets.*

Lemma 1.5.4. *[29, Lemma 9.7] An arbitrary intersection and a finite union of multicomplexes is again multicomplex.*

Corollary 1.5.5. *[29, Corollary 9.8] Let $A \subset \mathbb{N}^n_\infty$ be an arbitrary subset of $\mathbb{N}^n_\infty$. Then there exists a unique smallest multicomplex $\Gamma(A)$ containing A. In particular if $A = \{\mathbf{a}\}$, then*

$$\Gamma(\mathbf{a}) = \{\mathbf{b} \in \mathbb{N}^n_\infty \colon \mathbf{b} \leq \mathbf{a}\}.$$

For $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ we set $x^{\mathbf{a}} = \prod_{i=1}^n x_i^{a_i}$. Let Γ be a multicomplex, and let $I(\Gamma)$ be the K-vector space in $S = K[x_1, \ldots, x_n]$ spanned by all monomials $x^{\mathbf{a}}$ such that $\mathbf{a} \in \mathbb{N}^n \setminus \Gamma$. Note that if $\mathbf{a} \in \mathbb{N}^n$ and $\mathbf{b} \in \mathbb{N}^n \setminus \Gamma$, then $\mathbf{a} + \mathbf{b} \in \mathbb{N}^n \setminus \Gamma$. That is if $x^{\mathbf{b}} \in I(\Gamma)$, then $x^{\mathbf{a}} x^{\mathbf{b}} \in I(\Gamma)$ for all $\mathbf{a} \in \mathbb{N}^n$. Therefore $I(\Gamma)$ is a monomial ideal, and called the monomial ideal associated to Γ. conversely, let $I \subset S$ be any monomial ideal, then there exists a unique multicomplex $\Gamma(I)$ with $I(\Gamma(I)) = I$. Indeed, let $A = \{a \in \mathbb{N}^n : x^a \notin I\}$; then $\Gamma(I) = \Gamma(A)$ is called the multicomplex associated to I , where $\Gamma(A)$ is the unique smallest multicomplex containing A.

The monomial ideals and multicomplexes behave with respect to intersection and union as in the following:

Lemma 1.5.6. *[29, Lemma 9.9] Let $\Gamma_j \quad j \in J$ be a family of multicomplexes. Then*

(i) *$I(\bigcap_{j \in J} \Gamma_j) = \sum_{j \in J} I(\Gamma_j)$;*

(ii) *if J is finite, then $I(\bigcup_{j \in J} \Gamma_j) = \bigcap_{j \in J} I(\Gamma_j)$.*

Let Γ be a multicomplex with just one maximal facet $\mathbf{a}$. Then

$$I(\Gamma) = (x_i^{a_i+1} \colon i \in [n] \setminus \operatorname{infpt} \mathbf{a}),$$

see [29, Lemma 9.10]. In particular $\mathbf{a}$ is the only facet of Γ if and only if $\mathbf{a} \in \{0, \infty\}^n$ and $I(\Gamma) = P_{\mathbf{a}}$, where

$$P_{\mathbf{a}} = (x_i \colon i \notin \operatorname{infpt} \mathbf{a}).$$

Proposition 1.5.7. *[29, Proposition 9.12] Let* Γ *be a multicomplex with maximal facets* $\mathbf{a}_1, \cdots, \mathbf{a}_r$*. Then*

$$I(\Gamma) = \bigcap_{i=1}^{r} I(\Gamma(\mathbf{a}_j)).$$

It is easy to see that if Γ is a multicomplex and $I(\Gamma)$ the monomial ideal associated with Γ, then $\dim S/I(\Gamma) = \dim \Gamma + 1$, see [29, Corollary9.13].

A subset $S \subset \mathbb{N}^n_\infty$ is called a *Stanley* set if there exists $a \in \mathbb{N}^n$ and $m \in \{0, \infty\}^n$ such that $S = a + S^*$, where $S^* = \Gamma(m)$.

In [29] the concept of *shelling* of multicomplexes was introduced as in the following.

Definition 1.5.8. A multicomplex Γ is *shellable* if the facets of Γ can be ordered $a_1, \ldots, a_r$ such that

(i) $S_i = \Gamma(a_i) \backslash \Gamma(a_1, \ldots, a_{i-1})$ is a Stanley set for all $i = 1, \ldots, r$, and

(ii) whenever ${S_i}^* \subseteq {S_j}^*$, then ${S_i}^* = {S_j}^*$ or $i > j$.

Any order of the facets satisfying (i) and (ii) is called a *shelling* of Γ.

In [29, Theorem 10.5] the following has been proved.

Theorem 1.5.9. *The multicomplex* Γ *is shellable if and only if* $S/I(\Gamma)$ *is a pretty clean* S*-module.*

The multicomplex Γ is called *Cohen–Mavaulay or sequentially Cohen–Macaulay* if $S/I(\Gamma)$ is Cohen–Macaulay or sequentially Cohen–Macaulay.

Corollary 1.5.10. *If* Γ *is a shellable multicomplex, then* Γ *is sequentially Cohen–Macaulay. Moreover if all facets of* Γ *have the same dimension, then* Γ *is Cohen–Macaulay.*

We conclude this section with the following corollary [29, Corollary 10.7].

Corollary 1.5.11. *A multicomplex* Γ *is shellable if and only if there exists an order* $\mathbf{a}_1, \ldots, \mathbf{a}_r$ *of the facets of* Γ *such that for* $i = 1, \ldots, r$ *the sets*

$$S_i = \Gamma(\mathbf{a}_i) \setminus \Gamma(\mathbf{a}_1, \ldots, \mathbf{a}_{i-1})$$

are Stanley sets with $\dim S_1 \geq \dim S_2 \geq \ldots, \geq \dim S_r$*.*

1.6 Squarefree modules

In this section we collect some facts about squarefree modules, which are a natural extension of the concept of Stanley–Reisner rings associated to simplicial complexes. We fix some notation and recall some definitions. For $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, we say $\mathbf{a}$ is squarefree if $a_i = 0$ or $a_i = 1$ for $i = 1, \ldots, n$. We set $\operatorname{supp}(\mathbf{a}) = \{i\colon a_i \neq 0\} \subset [n] = \{1, \ldots, n\}$ and $|\mathbf{a}| = \sum_{i=1}^{n} a_i$. Occasionally we identify a squarefree vector $\mathbf{a}$ with $\operatorname{supp}(\mathbf{a})$. Let $\varepsilon_i = (0 \ldots, 1, \ldots, 0) \in \mathbb{N}^n$ be the vector with 1 at the i-th position. Let $M = \bigoplus_{\mathbf{a}\in\mathbb{Z}^n} M_{\mathbf{a}}$ be an $\mathbb{Z}^n$-graded K-vector space. For simplicity set $\operatorname{supp}(m) = \operatorname{supp}(\deg m)$ and $|m| = |\deg m|$ for any homogeneous element $m \in M$. A homogeneous element $m \in M$ is called *squarefree* if $\deg m \in \{0, 1\}^n$.

Let K be a field and $S = K[x_1, \ldots, x_n]$ the symmetric algebra over K. Consider the natural $\mathbb{N}^n$-grading on S. For a monomial $x_1^{a_1} \cdots x_n^{a_n}$ with $\mathbf{a} = (a_1, \ldots, a_n)$ we set $x^{\mathbf{a}}$, and for $F \subset [n]$ we denote $x_F = \prod_{j\in F} x_j$.

Let V be an n-dimensional K-vector space with basis $e_1, \ldots, e_n$. We denote by $E = K\langle e_1, \ldots, e_n\rangle$ the exterior algebra over V. The algebra E is a naturally $\mathbb{N}^n$-graded K-algebra with $\deg e_i = \varepsilon_i$. Let $F = \{j_1 < j_2 < \ldots < j_k\} \subset [n]$. Then $e_F = e_{j_1} \wedge e_{j_2} \wedge \ldots \wedge e_{j_k}$ is called a monomial in E. It is easy to see that the elements e_F, with $F \subset [n]$ form a K-basis of E. Here we set $e_F = 1$, if $F = \emptyset$. For any $\mathbf{a} \in \mathbb{N}^n$ we set $e_{\mathbf{a}} = e_{\operatorname{supp}(\mathbf{a})}$ and $\operatorname{supp}(e_{\mathbf{a}}) = \operatorname{supp}(\mathbf{a})$. For monomials $u, v \in E$ with $\operatorname{supp}(u) \subset \operatorname{supp}(v)$ there exists a unique term $w \in E$ such that $u \wedge w = v$. We set $w = u^{-1} \wedge v$. Let $u, v, w, z \in E$ be monomials. The equalities below holds whenever the expressions are defined:

$$(v^{-1} \wedge u) \wedge w = v^{-1} \wedge (u \wedge w) \quad \text{and} \quad (z^{-1} \wedge v) \wedge (v^{-1} \wedge u) = z^{-1} \wedge u.$$

A finite dimensional K-vector space M is called an $\mathbb{Z}^n$-graded E-module, if

(i) $M = \bigoplus_{\mathbf{a}\in\mathbb{N}^n} M_{\mathbf{a}}$ is a direct sum of K-vector spaces $M_{\mathbf{a}}$;

(ii) M is an $(E - E)$-bimodule;

(iii) for all vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{Z}^n$ and all $f \in E_{\mathbf{a}}$ and $m \in M_{\mathbf{b}}$ one has $fm \in M_{\mathbf{a}+\mathbf{b}}$ and $fm = (-1)^{|\mathbf{a}||\mathbf{b}|} mf$.

The following definition is due to Yanagawa [56].

Definition 1.6.1. A finitely generated $\mathbb{N}^n$-graded S-module $M = \bigoplus_{\mathbf{a}\in\mathbb{N}^n} M_{\mathbf{a}}$ is *squarefree* if the multiplication map $M_{\mathbf{a}} \to M_{\mathbf{a}+\varepsilon_i}$, $m \mapsto mx_i$, is bijective for all $\mathbf{a} \in \mathbb{N}^n$ and all $i \in \operatorname{supp}(\mathbf{a})$.

For example the Stanley-Reisner ring $K[\Delta]$ of a simplicial complex Δ is a squarefree S-module. If $I \subset J$ are squarefree monomial ideals, then I, S/I and J/I are squarefree S-modules. The following example shows that the factor module J/I may be a squarefree $\mathbb{N}^n$-graded S-module, even though I and J are not squarefree monomial ideals.

Example 1.6.2. Let $I = (x^2, xy) \subset J = (x^2, xy, yz)$ be monomial ideals in $K[x, y, z]$. Let $u \in S$ be a monomial. Then $u \in J \setminus I$ if and only if $u = (yz)v$ for some $v \in K[y, z]$. Hence J/I is a squarefree $\mathbb{N}^n$-graded S-module. But if we choose $I' = (x^2, yz) \subset J = (x^2, xy, yz) \subset K[x, y, z]$, then $xy \in J \setminus I'$ and $x(xy) = x^2y \in I'$. Therefore J/I' is not a squarefree $\mathbb{N}^n$-graded S-module.

Since $\dim_K (J/I)_{\mathbf{a}} \leq 1$ for all $\mathbf{a} \in \mathbb{N}^n$, the $\mathbb{N}^n$-graded S-module J/I is squarefree if and only if the multiplication map

$$(J/I)_{\mathbf{a}} \to (J/I)_{\mathbf{a}+\varepsilon_i},\ m \to x_i m$$

is injective for all $i \in \operatorname{supp}(m)$ and all $\mathbf{a} \in \mathbb{N}^n$.

Remark 1.6.3. Let $I \subset J \subset S$ be two monomial ideals. The $\mathbb{N}^n$-graded S-module J/I is squarefree if and only all minimal monomial generators of J/I are squarefree monomials and $\operatorname{supp}(u) \not\subset \operatorname{supp}(m)$ for all $m \in J \setminus I$ and all $u \in G(I)$ where $G(I)$ denote the set of minimal monomial generators of I. Indeed let J/I be a squarefree S-module and one of the minimal generators of J/I is not squarefree, say $m \in J \setminus I$. We may assume that $x_1^2 \mid m$ and $\deg(m) = \mathbf{a}$. Then $m' = m/x_1 \in (J/I)_{\mathbf{a}-\varepsilon_1}$ is a zero element and $1 \in \operatorname{supp}(m')$ but $m = x_1 m' \in (J/I)_{\mathbf{a}}$ is a non-zero element, a contradiction. Also if there exists a monomial $m \in J \setminus I$ and a monomial $u \in G(I)$ such that $\operatorname{supp}(u) \subset \operatorname{supp}(m)$, then we can find a minimal monomial $m' = mx^{\mathbf{a}}$ (with respect to divisibility) such that $\operatorname{supp}(\mathbf{a}) \subset \operatorname{supp}(m)$, $u \mid m'$ and $m'/x_i \notin I$ for some $i \in \operatorname{supp}(\mathbf{a})$, again a contradiction.

For the converse assume that J/I is minimally generated by squarefree monomials in $J \setminus I$ and $\operatorname{supp}(u) \not\subset \operatorname{supp}(m)$ for all $m \in J \setminus I$ and for all $u \in G(I)$. Let $m \in S$ be a monomial and $i \in \operatorname{supp}(m)$. Since the minimal monomial generators of J/I are squarefree, if $m \notin J$, then $x_i m \notin J$ or $x_i m \in J \cap I$. Hence in this case the multiplication map $m \to x_i m$ is injective. In the case that if $m \in J \setminus I$, then $x_i m \notin I$. Otherwise there must exist a $u \in G(I)$ such that $u \mid x_i m$. Therefore $\operatorname{supp}(u) \subset \operatorname{supp}(x_i m) = \operatorname{supp}(m)$ which is a contradiction.

Yanagawa [56, Lemma 2.3] proved that if M and M' are squarefree S-modules and $\varphi\colon M \to M'$ is a $\mathbb{N}^n$-homogeneous homomorphism, then $\operatorname{Ker}\varphi$ and $\operatorname{Coker}\varphi$ are again squarefree S-modules. This implies that each syzygy module $\operatorname{Syz}_i(M)$ in a multigraded minimal free S-resolution $F_\bullet$ of M is squarefree.

It is easy to see that if M is a squarefree S-module, then $\dim_K M_{\mathbf{a}} = \dim_K M_{\operatorname{supp}(\mathbf{a})}$ for any $\mathbf{a} \in \mathbb{N}^n$, and M is generated by its squarefree parts $\{M_F\colon F \subset [n]\}$.

Remark 1.6.4. Let M be a squarefree S-module and $F_\bullet$ be the minimal $\mathbb{N}^n$-graded free S-resolution of M. Then the $\mathbb{N}^n$-graded free S-module F_i is generated by squarefree elements. We call $F_\bullet$ a *squarefree* resolution of M. It is easy to see that an $\mathbb{N}^n$-graded S-module M is squarefree if and only if it has a squarefree resolution.

Let $SQ(S)$ be the abelian category of the squarefree S-modules where the morphisms are the $\mathbb{N}^n$-graded homogeneous homomorphisms. The following construction, which is of crucial important for this chapter, is due to Aramova, Avramov and Herzog [1].

Construction 1.6.5. Let (F, θ) be a squarefree complex of $\mathbb{N}^n$-graded S-modules, meaning that each F_i has a basis B_i with $\deg(f) \in \mathbb{N}^n$ is squarefree for all $f \in B_i$.

For $\mathbf{a} \in \mathbb{N}^n$ and $f \in B_i$ let $y^{(\mathbf{a})}f$ be a symbol to which we assign $\deg(y^{(\mathbf{a})}f) = \mathbf{a} + \deg(f)$. Now we define the $\mathbb{N}^n$-graded free E-module G_l with basis $y^{(\mathbf{a})}f$ where $\mathbf{a} \in \mathbb{N}^n$, $f \in B_i$, $\operatorname{supp}(\mathbf{a}) \subset \operatorname{supp}(f)$ and $l = |\mathbf{a}| + i$. For $f \in B_i$ and

$$\theta_i(f) = \sum_{f_j \in B_{i-1}} t_j x^{\mathbf{c}-\mathbf{c}_j} \quad \text{with} \quad t_j \in K, \quad \mathbf{c} = \deg(f), \quad \mathbf{c}_j = \deg(f_j),$$

we define homomorphisms $G_l \to G_{l-1}$ of $\mathbb{N}^n$-graded E-modules by

$$\gamma_l(y^{(\mathbf{a})}f) = (-1)^{|\mathbf{c}|} \sum_{k \in \operatorname{supp}(\mathbf{a})} y^{(\mathbf{a}-\varepsilon_k)} f e_k$$

$$\varphi_l(y^{(\mathbf{a})}f) = (-1)^{|\mathbf{a}|} \sum_{f_j \in B_{i-1}} y^{(\mathbf{a})} f_j t_j e_{\mathbf{c}_j}^{-1} e_{\mathbf{c}}.$$

Now we set $\sigma_l = \gamma_l + \varphi_l|\colon G_l \to G_{l-1}$. Then (G, σ) is a complex of free $\mathbb{N}^n$-graded E-modules. If (G', σ') is another complex obtained by different homogeneous basis B' of (F, θ), then (G, σ) and $(G;, \sigma')$ are isomorphic as complexes of $\mathbb{N}^n$-graded E-modules.

The following result is important for us.

Theorem 1.6.6. *[39, Theorem 1.2] If $(F_\bullet, \theta)$ is the minimal free $\mathbb{N}^n$-graded S-resolution of a squarefree S-module M, then $(G_\bullet, \sigma)$ is the minimal free $\mathbb{N}^n$-graded E-resolution of $N := \operatorname{Coker}(G_1 \to G_0)$.*

Next we recall the following definition which is due to T. Römer [39].

Definition 1.6.7. A finitely generated $\mathbb{N}^n$-graded E-module $N = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} N_{\mathbf{a}}$ is called squarefree if it has only squarefree components.

Let Δ be a simplicial complex on vertex set $[n]$. Then $K\{\Delta\} = E/J_\Delta$ is called the exterior face ring of Δ, where $J_\Delta = (e_{i_1} \wedge \cdots \wedge e_{i_k} \colon \{i_1, \ldots, i_k\} \notin \Delta)$. Notice that $K\{\Delta\} = E/J_\Delta$ is $\mathbb{N}^n$-graded E-module.

Observe that for a homogeneous homomorphism $\varphi\colon N \to N'$ between two squarefree E-modules, one has $\operatorname{Ker} \varphi$ and $\operatorname{Coker} \varphi$ are again squarefree E-modules. We denote by $SQ(E)$ the abelian category of squarefree E-modules where the morphisms are the $\mathbb{N}^n$-graded homogeneous homomorphisms.

The following construction is due to Tim Römer which is inverse to 1.6.5.

Construction 1.6.8. Let N be a squarefree E-module and $(G_\bullet, \sigma)$ be the minimal $\mathbb{N}^n$-graded free E-resolution of N. Let B_i be a homogeneous basis of G_i for all $i \in \mathbb{N}$. We set $\tilde{B}_i = \{f \in B_i \colon \deg(f) \quad \text{is squarefree}\}$. Then we define a complex (F, θ) of S-modules where F_i is the $\mathbb{N}^n$-graded free S-module with basis $\tilde{B}_i$. If $f \in \tilde{B}_i$ and

$$\sigma_i(f) = \sum_{f_j \in \tilde{B}_{i-1}} t_j f_j e_{\mathbf{b}_j}^{-1} e_{\mathbf{b}} \quad \text{with} \quad t_j \in K \quad \mathbf{b} = \deg(f), \quad \mathbf{b}_j = \deg(f_j),$$

we set $\theta_i(f) = \sum_{f_j \in \tilde{B}_{i-1}} t_j f_j x^{\mathbf{b}-\mathbf{b}_j}$.

Theorem 1.6.9. *Let N be a squarefree E-module and $(G_\bullet, \sigma)$ be the minimal $\mathbb{N}^n$-graded free E-resolution of N. The constructed complex (F, θ) is the minimal $\mathbb{N}^n$-graded free S-resolution of $M := \operatorname{Coker}(F_1 \to F_0)$ and $M \in SQ(S)$.*

Römer [39, Corollary 1.6] proved that there are two exact additive covariant functors

$$\mathbf{F}\colon SQ(S) \mapsto SQ(E),\ \ M \mapsto \mathbf{F}(M) \quad \text{and} \quad \mathbf{G}\colon SQ(E) \mapsto SQ(S),\ \ N \mapsto \mathbf{G}(N)$$

of abelian categories such that $(\mathbf{F} \circ \mathbf{G})(N) = N$ and $(\mathbf{G} \circ \mathbf{F})(M) = M$. Hence the categories $SQ(S)$ and $SQ(E)$ are equivalent. For example if $\Gamma \subset \Delta$ are simplicial complexes, then $\mathbf{F}(I_\Gamma/I_\Delta) = J_\Gamma/J_\Delta$ and $\mathbf{G}(J_\Gamma/J_\Delta) = I_\Gamma/I_\Delta$.

Let $M \in SQ(S)$. By the construction of $N = \mathbf{F}(M)$ given in 1.6.5, each minimal homogeneous system of generators $m_1, \ldots, m_t$ of M corresponds to a minimal homogeneous system of generators $n_1, \ldots, n_t$ of $N = \mathbf{F}(M)$, and for all $F \subset [n]$ we have an isomorphism of K-vector spaces $\theta_F\colon M_F \to \mathbf{F}(M)_F$. This isomorphism is described as follows: an element $m \in M_F$ can be written as $m = \sum a_i m_i x_{F_i}$, where $a_i \in K$ and where F is the disjoint union of F_i and $\deg(m_i) = G_i$ for each i. Then

$$\theta_F(m) = \sum (-1)^{\sigma(G_i, F_i)} a_i n_i e_{F_i}, \tag{1}$$

where $\sigma(G_i, F_i) = |\{(r, s)\colon r \in G_i,\ s \in F_i,\ r > s\}|$. The definition of θ_F does not depend on the particular presentation of m as a homogeneous linear combination of the m_i. In particular, we have that $\theta_{G_i}(m_i) = n_i$ for all i.

We set $M_{\mathrm{sq}} = \bigoplus_F M_F$ and define the isomorphism of graded K-vector spaces $\theta\colon M_{\mathrm{sq}} \to N$ by requiring that $\theta(m) = \theta_F(m)$ for all $m \in M_F$. Now Formula (1) can be extended as follows:

Lemma 1.6.10. *Let m be a squarefree element of M with $\operatorname{supp}(m) = F$, and let $m = \sum_i a_i w_i x_{L_i}$ with $a_i \in K$ and w_i squarefree with $\operatorname{supp}(w_i) = F_i$ such that F is the disjoint union of F_i and L_i for all i. Then*

$$\theta(m) = \sum a_i (-1)^{\sigma(F_i, L_i)} \theta(w_i) e_{L_i}.$$

Proof. Let $m_1, \ldots, m_t$ be a minimal homogeneous system of generators of M and let $n_1, \ldots, n_t$ be the corresponding minimal homogeneous system of generators of N with $\theta(m_i) = n_i$. Let $w_i = \sum b_{ij} m_{ij} x_{H_{ij}}$ where $b_{ij} \in K$ and where F_i is a disjoint union of $G_{ij} = \operatorname{supp}(m_{ij})$ and H_{ij} for all ij. Then

$$\begin{aligned}
\theta(m) &= \theta(\sum_i a_i (\sum_j b_{ij} m_{ij} x_{H_{ij}}) x_{L_i} = \theta(\sum_i \sum_j a_i b_{ij} m_{ij} x_{H_{ij} \cup L_i}) \\
&= \sum_i \sum_j (-1)^{\sigma(G_{ij}, H_{ij} \cup L_i)} a_i b_{ij} n_{ij} e_{H_{ij} \cup L_i}.
\end{aligned}$$

On the other hand

$$\begin{aligned}\sum \quad & a_i(-1)^{\sigma(F_i,L_i)}\theta(w_i)e_{L_i} = \sum_i\sum_j(-1)^{\sigma(G_{ij}\cup H_{ij},L_i)}(-1)^{\sigma(G_{ij},H_{ij})}a_ib_{ij}n_{ij}e_{H_{ij}}e_{L_i}\\
= \quad & \sum_i\sum_j(-1)^{\sigma(G_{ij},L_i)}(-1)^{\sigma(H_{ij},L_i)}(-1)^{\sigma(G_{ij},H_{ij})}(-1)^{\sigma(H_{ij},L_i)}a_ib_jn_{ij}e_{H_{ij}\cup L_i}\\
= \quad & \sum_i\sum_j(-1)^{\sigma(G_{ij},L_i)}(-1)^{\sigma(G_{ij},H_{ij})}a_ib_jn_{ij}e_{H_{ij}\cup L_i}\\
= \quad & \sum_i\sum_{ij}(-1)^{\sigma(G_{ij},H_{ij}\cup L_i)}a_ib_{ij}n_{ij}e_{H_{ij}\cup L_i} = \theta(m).\end{aligned}$$

$\square$

In the category of squarefree E-modules the graded E-dual is defined to be $N^\vee = \mathrm{Hom}_E(N, E)$. Observe that $N^\vee$ is again a squarefree E-module and by [2, 5.1(a)] one has $()^\vee$ is an exact contravariant functor. Let $\Gamma \subset \Delta$ be two simplicial complexes. Then $(J_\Gamma/J_\Delta)^\vee = J_{\Delta^\vee}/J_{\Gamma^\vee}$, see [39, Lemma 1.8].

In [39], the Alexander dual of a squarefree S-module is defined as follows:

Definition 1.6.11. *Let $M \in SQ(S)$. Then $M^\vee = \mathbf{G}(\mathbf{F}(M)^\vee)$ is called the Alexander dual of M.*

Note that

$$SQ(S) \to SQ(S), \quad M \to M^\vee$$

is a contravariant exact functor. Notice that Alexander dual is also defined for any $\mathbb{N}^n$-graded S-module in general by Miller [35]. In squarefree case his definition coincides to Römer definition. In this thesis we use the approach of Römer.

For example if $I \subset J$ are squarefree monomial ideals in S. Let Δ and Γ be simplicial complexes with $I = I_\Delta$ and $J = I_\Gamma$. Then J/I is a squarefree S-module and $(J/I)^\vee = I_{\Delta^\vee}/I_{\Gamma^\vee}$. In particular if Δ is a simplicial complex on the vertex set $[n]$ and I_Δ its Stanley-Reisner ideal, then $(S/I_\Delta)^\vee = I_{\Delta^\vee}$ and $(I_\Delta)^\vee = S/I_{\Delta^\vee}$.

Let W be an $\mathbb{Z}^n$-graded K-vector space. Then $W^* = \mathrm{Hom}_K(W, K(-\mathbf{1}))$ is again a $\mathbb{Z}^n$-graded K-vector space with the graded components

$$(W^*)_{\mathbf{a}} = \mathrm{Hom}_K(W_{\mathbf{1}-\mathbf{a}}, K) \text{ for all } \mathbf{a} \in \mathbb{Z}^n.$$

Here $\mathbf{1} = (1, \dots, 1)$. Note that if W is an $\mathbb{Z}^n$-graded E-module, then W^* is also a $\mathbb{Z}^n$-graded E-module. Furthermore if W is a squarefree E-module, then W^* is again a squarefree E-module.

Let $\varphi \in N^\vee$ and $n \in N$. Then $\varphi(n) = \sum_{F\subseteq[n]} \varphi_F(n)e_F$ with $\varphi_F(n) \in K$ for all $F \subseteq [n]$. Therefore for each $F \subseteq [n]$ we obtain a K-linear map $\varphi_F : N \to K$.

The following theorem is important for us later.

Theorem 1.6.12. *[22] Let N be a $\mathbb{Z}^n$-graded E-module. The map $\eta\colon N^\vee \to N^*$, $\varphi \to \varphi_{[n]}$ is a functorial isomorphism of $\mathbb{Z}^n$-graded E-modules. In particular if N is squarefree E-module, then $N^\vee$ is again squarefree and η is a functorial isomorphism of squarefree E-modules.*

2 Clean and pretty clean K-algebras

We denote by $S = K[x_1, \dots, x_n]$ the polynomial ring in n variables over a field K. Let $I \subset S$ be a monomial ideal. In this chapter we study prime filtrations of K-algebras of the form S/I where $I \subset S$ is a monomial ideal. In this thesis a prime filtration of S/I is always assumed to be a monomial prime filtration, i.e. a prime filtration

$$\mathcal{F}: \ (0) = I_0/I \subset I_1/I \subset \cdots \subset I_r/I = S/I$$

where each I_j a monomial ideal. Such prime filtration is equivalent to a filtration

$$I = I_0 \subset I_1 \subset \cdots \subset I_r = S$$

of monomial ideals such that $I_j/I_{j-1} \cong S/P_j$ for all j, where P_j is a monomial prime ideal. Recall that the prime filtration $\mathcal{F}$ is called *pretty clean*, if for all $i < j$ which $P_i \subseteq P_j$ it follows that $P_i = P_j$. The monomial ideal I is called *pretty clean*, if S/I has a pretty clean filtration.

2.1 Pretty clean monomial ideals and multicomplexes

In this section we will show that monomial ideals in at most three variables are pretty clean. For this first we show that if $I \subset S$ is a monomial ideal with height $I \geq n-1$, then I is pretty clean. By an example we show that one can not extend these results for monomial ideals $I \subset S$ when $n \geq 4$ or height $I \leq n-2$, see Example 2.1.8.

Let $I \subset S$ be a monomial ideal. The saturation $\tilde{I}$ of I is defined to be

$$\tilde{I} = I : \mathfrak{m}^\infty = \bigcup_k (I : \mathfrak{m}^k),$$

where $\mathfrak{m} = (x_1, \dots, x_n)$ is the graded maximal ideal of S.

We first note the following

Lemma 2.1.1. *Let $I \subset S$ be a monomial ideal of S. The ideal I is pretty clean if and only if $\tilde{I}$ is pretty clean.*

Proof. The K-vector space $\tilde{I}/I$ has a finite dimension, and we can choose monomials $u_1, \dots, u_t \in \tilde{I}$ whose residue classes modulo I form a K-basis of $\tilde{I}/I$. Moreover the basis can be chosen such that for all $j = 1, \dots, t$ one has $I_j/I_{j-1} \cong S/\mathfrak{m}$ where $I_0 = I$ and $I_j = (I_{j-1}, u_j)$, and where $\mathfrak{m} = (x_1, \dots, x_n)$ is the graded maximal ideal of S. Indeed, we have $\tilde{I} = I : \mathfrak{m}^k$ for some k. For each $i \in [k]$, where $[k] = \{1, \dots, k\}$, the K-vector space $(I : \mathfrak{m}^i)/(I : \mathfrak{m}^{i-1})$ has finite dimension. If

$$\dim_K(I : \mathfrak{m}^i / I : \mathfrak{m}^{i-1}) = r_i,$$

then we can choose monomials $u_{i,1}, \dots, u_{i,r_i} \in I : \mathfrak{m}^i$ whose residue classes modulo $I : \mathfrak{m}^{i-1}$ form a basis for this K-vector space. Composing these bases we obtain the required basis for $\tilde{I}/I$.

So we have

$$\mathcal{F}_1 : \ I = I_0 \subset I_1 \subset \cdots \subset I_t = \tilde{I}$$

with $I_i/I_{i-1} \cong S/\mathfrak{m}$, for all $i = 1, \ldots, t$. Now if $\tilde{I}$ is pretty clean and $\mathcal{G}$ is the pretty clean filtration of $\tilde{I}$, then the prime filtration $\mathcal{F}$ which is obtained by composing $\mathcal{F}_1$ and $\mathcal{G}$ yields a pretty clean filtration of I.

For the converse, let $I = I_0 \subset I_1 \subset \cdots \subset I_r = S$ be pretty clean filtration of I. We will show that $\tilde{I}$ is pretty clean by induction on $\dim_K \tilde{I}/I = t$. If $t = 0$ the assertion is trivially true. Assume now that $t > 0$. It is clear that I_1 is also pretty clean and that $I_1/I \cong S/\mathfrak{m}$, since $I \neq \tilde{I}$. It follows that $\tilde{I_1} = \tilde{I}$ and that $\dim_K \tilde{I_1}/I_1 = t - 1$. So by the induction hypothesis $\tilde{I} = \tilde{I_1}$ is pretty clean. □

Corollary 2.1.2. *If $S = K[x_1, \ldots, x_n]$ is the polynomial ring in n variables, then any monomial ideal in S of height n is pretty clean.* □

Our next goal is to show that even the monomial ideals in $S = K[x_1, \ldots, x_n]$ of height at least $n - 1$ are pretty clean. To end this we show that the multicomplexes associated to them are shellable.

Remark 2.1.3. Let $\Gamma \subset \mathbb{N}^n_\infty$ be a shellable multicomplex with shelling $a_1, \ldots, a_r$, then $a_1(i) \in \{0, \infty\}$ and therefore a_1 is one of the minimal elements in $F(\Gamma)$ with respect to its partially order. Indeed, since $a_1, \ldots, a_r$ is a shelling, it follows that $S_1 = \Gamma(a_1)$ is a Stanley set and therefore there exists a vector $b \in \mathbb{N}^n$ and a vector $m \in \{0, \infty\}^n$ such that

$$\Gamma(a_1) = b + \Gamma(m).$$

It is clear that $\operatorname{infpt}(a_1) = \operatorname{infpt}(m)$. If $\operatorname{infpt}(m) = [n]$, then there is nothing to show. Suppose now that $\operatorname{infpt}(m) \neq [n]$, and choose $i \in [n] \setminus \operatorname{infpt}(m)$. If $a_1(i) \neq 0$ there exists $c \in \Gamma(a_1)$ with $c(i) < a_1(i)$. Since c and $a_1 \in b + \Gamma(m) = \Gamma(a_1)$, and since $m(i) = 0$, it follows that $c(i) = b(i) = a_1(i)$, a contradiction.

Furthermore, if Γ has only one maximal facet, then $F(\Gamma)$ has only one minimal element, also any shelling of Γ must start with this minimal element and end by the maximal one. In fact, suppose a_1 and a_2 are minimal elements in $F(\Gamma)$. By the first part of this remark it follows that a_1 and a_2 are vectors in $\{0, \infty\}^n$. Hence since $\operatorname{infpt}(a_1) = \operatorname{infpt}(a_2)$, we see that $a_1 = a_2$. Now let $a_1, \ldots, a_r$ be any shelling of Γ. Then, by what we have shown, it follows that a_1 is the unique minimal element in $F(\Gamma)$. Let m be the maximal element of $F(\Gamma)$. Suppose $m = a_k$ for some $k < r$, then

$$S_{k+1} = \Gamma(a_{k+1}) \setminus \Gamma(a_1, \ldots, a_k) = \Gamma(a_{k+1}) \setminus \Gamma(m) = \emptyset,$$

which is not a Stanley set, a contradiction. Moreover in this case for each i there exists a $d_i \in \mathbb{N}^n$ such that $S_i = d_i + \Gamma(a_1)$.

Now we show that in $S = K[x_1, \ldots, x_n]$, any ideal of height $n - 1$ is pretty clean.

Proposition 2.1.4. *If $I \subset S = K[x_1, \dots, x_n]$ is any monomial ideal of height $\geq n-1$, then I is pretty clean.*

Proof. We may assume that I is a monomial ideal of height $n-1$, and by Lemma 2.1.1 that I is saturated, i.e, $I = \tilde{I}$. It follows that $I = \bigcap I_j$, where

$$I_j = ({x_1}^{c_{j,1}}, \dots, x_{j-1}^{c_{j,j-1}}, x_{j+1}^{c_{j,j+1}}, \dots, {x_n}^{c_{j,n}})$$

and where $c_{j,k} > 0$ for $k \neq j$. We denote by Γ and Γ_j the multicomplexes associated to I and I_j, and by F and F_j the sets of facets of Γ and Γ_j, respectively. The sets F and F_j are finite, see [29, Lemma 9.6]. Suppose $|F| = t$ and $|F_j| = t_j$. Since I_j is P_j-primary where $P_j = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$, it follows from [29, Proposition 5.1] that I_j is pretty clean, and hence Γ_j is shellable. Moreover $a \in \mathbb{N}^n_\infty$ is a facet of Γ_j if and only if $a(j) = \infty$ and $a(k) < c_{j,k}$ for $k \neq j$. Let $a_{j,1}, \dots, a_{j,t_j}$ be a shelling of Γ_j.

For showing I is pretty clean it is enough to show that Γ is shellable. By [29, Lemma 9.9 (b)] we have $\Gamma = \bigcup_{j=1}^n \Gamma_j$. Also by [29, Lemma 9.10], each F_j has only one maximal facet, say m_j, where

$$m_j(k) = \begin{cases} \infty, & \text{if} \quad k = j, \\ c_{j,k} - 1, & \text{otherwise.} \end{cases}$$

It follows that $F = \bigcup F_j$ and that the union is disjoint, since $a \in F$ belongs to F_j if and only if $a(j) = \infty$ and $a(k) < \infty$ for $k \neq j$. In particular one has $(\bigcup_{i=1}^{j-1} F_i) \cap F_j = \emptyset$ for $j = 2, \dots, n$.

We claim that

$$a_{1,1}, \dots, a_{1,t_1}, a_{2,1}, \dots, a_{2,t_2}, \dots, a_{n,1}, \dots, a_{n,t_n}$$

is a shelling for Γ. Indeed, for all j and all k with $1 < k \leq t_j$ we have

$$S_{j,k} = \Gamma(a_{j,k}) \setminus \Gamma(a_{1,1}, \dots, a_{j,k-1}) = \Gamma(a_{j,k}) \setminus \Gamma(a_{j,1}, \dots, a_{j,k-1}),$$

and if $k = 1$, then

$$S_{j,1} = \Gamma(a_{j,1}) \setminus \Gamma(a_{1,1}, \dots, a_{j-1,t_{j-1}}) = \Gamma(a_{j,1}).$$

Since $a_{j,1}, \dots, a_{j,t_j}$ is a shelling of Γ_j, it follows that $S_{j,k}$ is a Stanley set for all j and all k.

Condition (ii) in the definition of shellability is obviously satisfied. In fact, since Γ_j is shellable and has only one maximal facet, it follows by Remark 2.1.3 that for all $k = 1, \dots, t_j$, there exists some $d_{j,k} \in \mathbb{N}^n$ such that $S_{j,k} = d_{j,k} + S_j^*$, where $S_j^* = \Gamma(a_{j,1})$. Moreover if $j \neq t$ then $a_{j,1}$ and $a_{t,1}$ are not comparable, and hence in this case there is no inclusion among S_j^* and S_t^*. $\square$

As a consequence of Proposition 2.1.4 we have

Corollary 2.1.5. *Any monomial ideal $I \subset S = K[x, y]$ is pretty clean.* □

Next we will show that any monomial ideal in $S = K[x_1, x_2, x_3]$ is also pretty clean. First we need

Lemma 2.1.6. *If $I \subset S = K[x_1, \dots, x_n]$ is a monomial ideal of height 1, then $I = uJ$, where u is a monomial in S, and J is a monomial ideal of height ≥ 2. Moreover, I is pretty clean if and only if J is pretty clean.*

Proof. The first statement of the lemma is obvious. Assume now that J is pretty clean with pretty clean filtration

$$\mathcal{F}: \; J = J_0 \subset J_1 \subset \cdots \subset J_r = S$$

such that $J_i/J_{i-1} \cong S/P_i$, where $P_i \in \operatorname{Ass} J$. Then height $P_i \geq 2$. It follows that

$$\mathcal{F}_1: \; I = uJ \subset uJ_1 \subset \cdots \subset uJ_r = (u)$$

is a prime filtration of $(u)/I$ with factors $uJ_i/uJ_{i-1} \cong S/P_i$.

There exists a prime filtration

$$\mathcal{F}_2: \; (u) = J_r \subset J_{r+1} \subset \cdots \subset J_{r+t} = S$$

of the principal monomial ideal $I_1 = (u)$, where the J_{r+k} are again principal monomial ideals with $J_{r+k}/J_{r+k-1} \cong S/Q_k$ and where $Q_k \in \operatorname{Ass}(u)$ has height 1 for all k. In fact, if $u = u_0 = \prod_{t=1}^{k} x_{i_t}^{a_t}$ and $u_j = \prod_{r=j+1}^{k} x_{i_r}^{a_r}$ for $j = 1, \dots, k-1$, Where $k \leq n$, then the prime filtration $\mathcal{F}_2$ is the following:

$$\begin{aligned}
\mathcal{F}_2: \; J_r &= (u) \subset (x_{i_1}^{a_1-1}u_1) \subset (x_{i_1}^{a_1-2}u_1) \cdots \subset (u_1) \subset (x_{i_2}^{a_2-1}u_2) \\
&\subset \cdots \subset (u_2) \subset \cdots \subset (x_{i_k}) \subset S.
\end{aligned}$$

Therefore this filtration of $I_1 = (u)$ is pretty clean. Now composing the above filtrations $\mathcal{F}_1$ and $\mathcal{F}_2$ we obtain a pretty clean filtration of I.

For the converse let I be pretty clean. Then from Corollary 2.2.12 we can choose a pretty clean filtration $\mathcal{F}$ for I with the property that the associated prime ideals of I with height one appear in the end of filtration. So we have

$$\mathcal{F}: \; I = I_0 = uJ \subset I_1 \subset \cdots \subset I_t = (u) \subset \cdots \subset I_r = S,$$

where $I_k = uJ_k$ for $k = 0, \dots, t$, and where $J_0 = J$. Since $J_i/J_{i-1} \cong I_i/I_{i-1}$ it follows that J is pretty clean with the pretty clean filtration $\mathcal{F}_1$, where

$$\mathcal{F}_1: \; J = J_0 \subset J_1 \subset \cdots \subset J_t = S.$$

□

Combining Lemma 2.1.6 with Proposition 2.1.4 we get

Theorem 2.1.7. *Any monomial ideal in a polynomial ring in at most three variables is pretty clean.*

The following example shows that this theorem can not be extended to polynomial rings in more than three variables, and it also shows that monomial ideals of height $< n-1$ may not be pretty clean.

Example 2.1.8. Let $n = 4$, and Γ be the multicomplex with facets $(\infty, \infty, 0, 0)$ and $(0, 0, \infty, \infty)$. Then Γ is not shellable, and so the monomial ideal

$$I(\Gamma) = (x_1x_3, x_1x_4, x_2x_3, x_2x_4) \subset K[x_1, x_2, x_3, x_4]$$

is not pretty clean.

More generally, let $n > 3$ and $a = (0, 0, \infty, \dots, \infty)$ and $b = (\infty, \infty, 0, \dots, 0)$ be two elements in $\mathbb{N}^n_\infty$. Then $\Gamma = \Gamma(a, b)$ is not a shellable multicomplex, hence $I = (x_1, x_2) \cap (x_3, \dots, x_n)$ is a squarefree monomial ideal in $S = K[x_1, \dots, x_n]$ which is not clean.

2.2 Pretty clean monomial ideals and polarizations

In this section we consider polarizations of monomial ideals and of prime filtrations. Let $S = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over the field K, and $u = \prod_{i=1}^n x_i^{a_i}$ be a monomial in S. Then

$$u^p = \prod_{i=1}^n \prod_{j=1}^{a_i} x_{i,j} \in K[x_{1,1}, \dots, x_{1,a_1}, \dots, x_{n,1}, \dots, x_{n,a_n}]$$

is called the *polarization* of u.

Let I be a monomial ideal in S with monomial generators $u_1, \dots, u_m$. Then $(u_1^p, \dots, u_m^p)$ is called *a polarization of* I. Note that if $v_1, \dots, v_k$ is a another set of monomial generators of I and if T is the polynomial with sufficiently many variables $x_{i,j}$ such that all the monomials u_i^p and v_j^p belong to T, then

$$(u_1^p, \dots, u_m^p)T = (v_1^p, \dots, v_k^p)T.$$

Therefore we denote any polarization of I by I^p, since in a common polynomial ring extension all polarizations are the same, and we write $I^p = J^p$ if a polarization of I and a polarization of J coincide in a common polynomial ring extension.

Now let $I = (u_1, \dots, u_m) \subset S$ be a monomial ideal, and $u \in S$ a monomial. Furthermore let T be the polynomial ring in variables $x_{i,j}$ such that:

(1) for all $i \in [n]$ there exists $k_i \geq 1$ such that $x_{i,1}, \dots, x_{i,k_i}$ are in T,

(2) $I^p \subset T$, and $u^p \in T$.

We consider the K-algebra homomorphism

$$\pi : T \longrightarrow S, \quad x_{i,j} \mapsto x_i.$$

Then π is an epimorphism with $\pi(u^p) = u$ for all monomials $u \in S$, and u^p is the unique squarefree monomial in T of the form $\prod_{i=1}^{n}\prod_{j=1}^{t_i} x_{i,j}$ with this property. In particular, $\pi(I^p) = I$. We call π the specialization map attached with the polarization.

Remark 2.2.1. Let $I = (u_1, \dots, u_m) \subset S$ be a monomial ideal, and $u \in S$ a monomial. Then

(a) $I : u = (u_i/\gcd(u_i, u))_{i=1}^{m}$, and it is again a monomial ideal in S.

(b) $I : u$ is a prime ideal if and only if for each $i \in [m]$, there exists a $j \in [m]$ such that $u_j/\gcd(u_j, u)$ is a monomial of degree one, and $u_j/\gcd(u_j, u)$ divides $u_i/\gcd(u_i, u)$.

(c) Let $u = \prod_{i=1}^{n} x_i^{a_i}$ and $u_j = \prod_{i=1}^{n} x_i^{b_i}$. If $u_j/\gcd(u_j, u) = x_i$, then $b_i = a_i + 1$ and $b_t \leq a_t$ for all $t \neq i$. Therefore $u_j/\gcd(u_j, u) = x_i$ if and only if $u_j^p/\gcd(u_j^p, u^p) = x_{i,b_i}$.

Lemma 2.2.2. *Let $I = (u_1, \dots, u_m) \subset S$ be a monomial ideal and $u \in S$ a monomial. If $I^p : u^p$ is a prime ideal, then $I^p : u^p = (x_{i_1,j_1}, \dots, x_{i_k,j_k})$ with $i_r \neq i_s$ for $r \neq s$.*

Proof. Since $I^p : u^p$ is a monomial prime ideal in polynomial ring T it must be generated by variables. If $x_{i,j}$ and $x_{i,k}$ are two generators of $I^p : u^p$, then there exist $r_j \in [m]$, and $r_k \in [m]$ such that $x_{i,j} = u_{r_j}^p/\gcd(u_{r_j}^p, u^p)$ and $x_{i,k} = u_{r_k}^p/\gcd(u_{r_k}^p, u^p)$. It follows from Remark 2.2.1(c) that $j - 1 = k - 1$ is equal to the exponent of x_i in u. Hence $x_{i,j} = x_{i,k}$. $\square$

Lemma 2.2.3. *Let $I = (u_1, \dots, u_m) \subset S$ be a monomial ideal, and $u \in S$ a monomial in S. Then $I : u$ is a prime ideal if and only if $I^p : u^p$ is a prime ideal. In this case $I : u = \pi(I^p : u^p)$.*

Proof. Let $I : u$ be a prime ideal. We may assume that $I : u = (x_1, \dots, x_k)$ for some $k \in [n]$. Therefore for each $i \in [k]$ there exists some u_{j_i}, with $j_i \in [m]$, such that $x_i = u_{j_i}/\gcd(u_{j_i}, u)$ and for each $t \in [m]$, there exists $i_t \in [k]$, such that x_{i_t} divides $(u_t/\gcd(u_t, u))$. Therefore by Remark 2.2.1(c) we have $u_{j_i}^p/\gcd(u_{j_i}^p, u^p) = x_{i,t_i}$, where t_i is the exponent of x_i in u_{j_i} and $t_i - 1$ is the exponent of x_i in u.

Also for each $s \in [m]$, the monomial $u_s^p/\gcd(u_s^p, u^p)$ is divisible by one of these x_{i,t_i}, where $i \in [k]$. Indeed, since $I : u$ is a prime ideal there exists some $i \in [k]$ such that x_i divides $(u_s/\gcd(u_s, u))$, where $x_i = u_{j_i}/\gcd(u_{j_i}, u)$. Let $t_i - 1$ be the exponent of x_i in u. Then it follows that the exponent of x_i in u_s is $> t_i - 1$. Hence x_{i,t_i} divides $u_s^p/\gcd(u_s^p, u^p)$, and $I^p : u^p = (x_{1,t_1}, \dots, x_{k,t_k})$.

For the converse, let $I^p : u^p$ be a prime ideal. By Lemma 2.2.2 we may assume that $I^p : u^p = (x_{1,t_1}, \ldots, x_{k,t_k})$. This means that for each $i \in [k]$ there is a monomial u_{j_i} with $j_i \in [m]$ such that $x_{i,t_i} = u_{j_i}^p / \gcd(u_{j_i}^p, u^p)$ and for each $s \in [m]$, the squarefree monomial $u_s^p / \gcd(u_s^p, u^p)$ is divisible by one of these x_{i,t_i}. Therefore by Remark 2.2.1(c) we have $x_i = u_{j_i} / \gcd(u_{j_i}, u)$ for $i \in [k]$, and for each $s \in [m]$, one of these variables divides $u_s / \gcd(u_s, u)$. Hence $I : u = (x_1, \ldots, x_k)$. □

Let $I \subset S$ be a monomial ideal and

$$\mathcal{F} : \ I = I_0 \subset I_1 \subset \cdots \subset I_r = S$$

a filtration of S/I. We call r the *length of filtration* $\mathcal{F}$ and denote it by $\ell(\mathcal{F})$.

Assume now that for all j we have $I_{j+1} = (I_j, u_j)$ where $u_j \in S$ is a monomial. We will define *the polarization* $\mathcal{F}^p$ of $\mathcal{F}$ inductively as follow: set $J_0 = I^p$; assuming that J_i is already defined, we set $J_{i+1} = (J_i, u_i^p)$. So $J_i = (I^p, u_1^p, \cdots, u_i^p)$, and

$$\mathcal{F}^p : \ I^p = J_0 \subset J_1 \subset \cdots \subset J_r = T$$

is a filtration of T/I^p.

We have the following

Proposition 2.2.4. *Suppose $I \subset S$ is a monomial ideal, and*

$$\mathcal{F} : \ I = I_0 \subset I_1 \subset \cdots \subset I_r = S$$

a filtration of S/I as above. Then $\mathcal{F}$ is a prime filtration of S/I if and only if $\mathcal{F}^p$ is a prime filtration of T/I^p.

Proof. Let

$$\mathcal{F} : \ I = I_0 \subset I_1 \subset \cdots \subset I_r = S$$

be a prime filtration of S/I. We use induction on $r = \ell(\mathcal{F})$ the length of prime filtration. If $r = 1$, then I is a monomial prime ideal and $I^p = I$.

Let $r > 1$. Then $\mathcal{F}_1 : I_1 \subset \ldots \subset I_r = S$ is a prime filtration of S/I_1, and $\ell(\mathcal{F}_1) = r - 1$. By our induction hypothesis, $\mathcal{F}_1^p$ is a prime filtration of $I_1^p = (I^p, u_1^p)$. Since $I_1/I \cong I_1 : u_1$ is a prime ideal, it follows from Lemma 2.2.3 that $J_0/J_1 \cong I_1^p : u_1^p$ is a prime ideal too. Hence $\mathcal{F}^p$ is a prime filtration of T/I^p.

The other direction of the statement is proved similarly. □

Let $S = K[x_1, \ldots, x_n]$ a the polynomial ring, and $u, v \in S$ be monomials. We notice that

$$\operatorname{lcm}(u, v)^p = \operatorname{lcm}(u^p, v^p).$$

Therefore we have

Lemma 2.2.5. *Let I, J be two monomial ideals in S. Then $(I \cap J)^p = I^p \cap J^p$.*

Proof. Let $I = (u_1, \dots, u_m)$ and $J = (v_1, \dots, v_t)$. Then $I \cap J = (\mathrm{lcm}(u_i, v_j))$, where $1 \le i \le m$ and $1 \le j \le t$. Therefore $(I \cap J)^p = (\mathrm{lcm}(u_i, v_j)^p) = (\mathrm{lcm}(u_i^p, v_j^p)) = I^p \cap J^p$. $\square$

We recall that a monomial ideal $I \subset S$ is an irreducible monomial ideal if and only if there exists a subset $A \subset [n]$ and for each $i \in A$ an integer $a_i > 0$ such that $I = (x_i^{a_i} \colon i \in A)$, see [54, Theorem 5.1.16]. It is known that for each monomial ideal I there exists a decomposition $I = \bigcap\limits_{i=1}^{r} J_i$ such that J_i are irreducible monomial ideals, see [54, Theorem 5.1.17].

Corollary 2.2.6. *Suppose $J_1, \dots, J_r$ are monomial ideals in the polynomial ring S, and $I = \bigcap\limits_{i=1}^{r} J_i$. Then $I^p = \bigcap\limits_{i=1}^{r} J_i^p$. In particular the minimal prime ideals of I^p are of the form $(x_{i_1,t_1}, \dots, x_{i_k,t_k})$, with $i_r \neq i_s$ for $r \neq s$.* $\square$

Next we show that if $I \subset S$ is a monomial ideal and I^p the polarization of I, then $|F(\Gamma(I))| = |F(\Gamma(I^p))|$. First we notice the following:

Lemma 2.2.7. *Let $I \subset S$ be an irreducible monomial ideal and I^p the polarization of I. Furthermore, let F and F^p be the sets of facets of $\Gamma(I)$ and $\Gamma(I^p)$, respectively. Then there exists a bijection between F and F^p.*

Proof. By [54, Theorem 5.1.16] there exists a subset $A \subset [n]$ and for each $i \in A$ an integer $a_i > 0$ such that $I = (x_i^{a_i} \colon i \in A)$. We may assume $A = [k]$ for some $k \le n$. In this case $\Gamma(I) = \Gamma(m)$, where

$$m(i) = \begin{cases} a_i - 1, & \text{if} \quad i \in [k], \\ \infty, & \text{otherwise}, \end{cases}$$

and $a \in F$ if and only if $a \le m$ and $a(i) = \infty$ for $i > k$. We have

$$I^p = (\prod_{j=1}^{a_1} x_{1,j}, \prod_{j=1}^{a_2} x_{2,j}, \dots, \prod_{j=1}^{a_k} x_{k,j}),$$

and we know that the facets in F^p correspond to the minimal prime ideals of I^p. Indeed, if $a \in F^p$ is a facet of Γ^p, then $P_a = (x_i \colon a(i) = 0)$ is a minimal prime ideal of I^p. Each minimal prime ideal of I^p is of the form $(x_{1,t_1}, \dots, x_{k,t_k})$, with $t_i \le a_i$.

Now we define

$$\theta : F \to F^p, \ a \mapsto \bar{a}$$

as follows: if $k < i \le n$, then $\bar{a}(ij) = \infty$ for all j, and if $i \in [k]$ we have $a(i) = t_i < a_i$, and we set

$$\bar{a}(ij) = \begin{cases} 0, & \text{if} \quad j = t_i + 1, \\ \infty, & \text{otherwise}. \end{cases}$$

Obviously $\bar{a} \in F^p$, since $P_{\bar{a}} = (x_{1,t_1+1}, \dots, x_{k,t_k+1})$ is a minimal prime ideal of I^p, and it is also clear that θ is an injective map.

Let $\bar{a} \in F^p$. Then $\bar{a}$ corresponds to the minimal prime ideal $P_{\bar{a}} = (x_{1,t_1}, \dots, x_{k,t_k})$, where $t_i \leq a_i$. Therefore if $k < i \leq n$, we have $\bar{a}(ij) = \infty$ for all j, and if $i \in [k]$, then

$$\bar{a}(ij) = \begin{cases} 0, & \text{if} \quad j = t_i, \\ \infty, & \text{otherwise.} \end{cases}$$

Let $a \in \mathbb{N}^n_\infty$ be the following:

$$a(i) = \begin{cases} t_i - 1, & \text{if} \quad i \in [k], \\ \infty, & \text{otherwise,} \end{cases}$$

then a is a facet in F, since $a \leq m$ and $\operatorname{infpt}(a) = n - k = \operatorname{infpt}(m)$, and moreover $\theta(a) = \bar{a}$. □

Now let $I = (u_1, \dots, u_m) \subset S$ be a monomial ideal and let $D \subset [n]$ be the set of elements $i \in [n]$ such that x_i divides u_j for at least one $j = 1, \dots, m$. Then we set

$$r_i = \max\{t\colon \; x_i^t \text{ divides } u_j \text{ at least for one } j \in [m]\}$$

if $i \in D$ and $r_i = 1$, otherwise. Moreover we set $r = \sum\limits_{i=1}^{n} r_i$.

Note that I has a decomposition $I = \bigcap\limits_{i=1}^{t} J_i$ where the ideals J_i are irreducible monomial ideals. In other words, each J_i is generated by pure powers of some of the variables. Then $I^p = \bigcap\limits_{i=1}^{t} J_i^p$ is an ideal in the polynomial ring

$$T = K[x_{1,1} \dots, x_{1,r_1}, x_{2,1} \dots, \dots, x_{n,1}, \dots, x_{n,r_n}]$$

in r variables.

We denote by Γ, Γ^p, Γ_i and Γ_i^p the multicomplexes associated to I, I^p, J_i and J_i^p, respectively, and by F, F^p, F_i and F_i^p the sets of facets of Γ, Γ^p, Γ_i and Γ_i^p, respectively.

It is clear that $F \subset \bigcup\limits_{i=1}^{t} F_i$ since $\Gamma = \bigcup\limits_{i=1}^{t} \Gamma_i$, and also that $F^p \subset \bigcup\limits_{i=1}^{t} F_i^p$. Each Γ_i has only one maximal facet, say m_i, and $m_i(k) \leq r_k - 1$ if $m_i(k) \neq \infty$.

Let $A \subset \mathbb{N}^n_\infty$ be the following set:

$$A = \{a \in \mathbb{N}^n_\infty \colon a(i) < r_i \quad \text{if} \quad a(i) \neq \infty\}.$$

We define the map

$$\beta : A \to \{0, \infty\}^r, \; a \mapsto \bar{a}$$

as follows: if $a(i)=\infty$, then $\bar{a}(ij)=\infty$ for all j, and if $a(i)=e$ where $e\leq r_i-1$, then

$$\bar{a}(ij)=\begin{cases}0, & \text{if} \quad j=e+1,\\ \infty, & \text{otherwise.}\end{cases}$$

Proposition 2.2.8. *With the above assumptions and notation the restriction of the map β to F is a bijection from F to F^p.*

Proof. First of all we want to show that $\bar{a}\in F^p$. Indeed, $a\in F\subset \bigcup_{i=1}^{t} F_i$. Therefore there exists an integer $j\in[n]$ such that $a\in F_j$, and since the restriction of β to F_j is the map θ defined in Lemma 2.2.7, it follows that $\bar{a}\in F_j^p$. Therefore there exists a subset $\{j_1,\ldots,j_s\}\subset[n]$ and positive integers t_k with $t_k\leq r_{j_k}$ for $k=1,\ldots,s$ such that $P_{\bar{a}}=(x_{j_1,t_1},\ldots,x_{j_s,t_s})$. It is clear that $P_{\bar{a}}$ is a prime ideal which contains I^p and $\beta(a)=\bar{a}$, where

$$a(i)=\begin{cases}t_k-1, & \text{if} \quad i=j_k \quad \text{for some } k,\\ \infty, & \text{otherwise.}\end{cases}$$

Now $\bar{a}\in F^p$ if and only if $P_{\bar{a}}\in \operatorname{Min}(I^p)$. Assume $P_{\bar{a}}\notin \operatorname{Min}(I^P)$. Then there is a prime ideal $Q\in\operatorname{Min}(I^p)$ such that $Q\subset P_{\bar{a}}$. Suppose $Q=(x_{i_1,e_1},\ldots,x_{i_h,e_h})$. Then $\{i_1,\ldots,i_h\}\subset\{j_1,\ldots,j_s\}$ and $\{e_1,\ldots,e_h\}\subset\{t_1,\ldots,t_s\}$. On the other hand, since Q is a minimal prime ideal of $I^p=\bigcap_{i=1}^t J_i^p$, there exists an integer $e\in[t]$ such that Q is one of the minimal prime ideals of

$$J_e^p=(x_{i_1}^{b_1},\ldots,x_{i_h}^{b_h})^p.$$

It follows that $1\leq e_i\leq b_i$ for $i=1,\ldots,h$. Therefore there exists $b\in F_e$ with

$$b(i)=\begin{cases}e_k-1, & \text{if} \quad i\in\{i_1,\ldots,i_h\},\\ \infty, & \text{otherwise.}\end{cases}$$

This implies that $a<b\leq m_e$, and $\operatorname{infpt}(a)<\operatorname{infpt}(b)=\operatorname{infpt}(m_e)$, a contradiction.

Next we show that β is injective: let $a,b\in F$ and $a\neq b$. Then there exists an integer i such that $a(i)\neq b(i)$. We have to show $\bar{a}\neq\bar{b}$. We consider different cases:

(i) If $a(i)=0$, and $b(i)\neq 0$, then $\bar{b}(i1)=\infty$ and $\bar{a}(i1)=0$.

(ii) If $a(i)=\infty$, and $b(i)=t-1$ where $t\neq\infty$, then $\bar{a}(it)=\infty$ and $\bar{b}(it)=0$.

(iii) Suppose $0<t-1=a(i)\neq\infty$. If $b(i)=0$, then we have case (i). If $b(i)=\infty$ then we have case (ii). Finally if $0<s-1=b(i)\neq\infty$, then $t\neq s$ since $a(i)\neq b(i)$ and hence $\bar{a}(it)=0$ and $\bar{b}(it)=\infty$.

In all cases it follows that $\bar{a}\neq\bar{b}$.

Finally we show that β is surjective: let $\bar{a}\in F^p\subset\bigcup_{i=1}^{t} F_i^p$ be any facet of Γ^p. Then there exists an integer $i\in[t]$ such that $\bar{a}\in F_i^p$. Therefore $P_{\bar{a}}$ is a minimal prime ideal of

$$J_i^p=(x_{i_1}^{a_1},\ldots,x_{i_k}^{a_k})^p,$$

and hence there exists $t_i \leq a_i$ such that $P_{\bar{a}} = (x_{i_1,t_1}, \ldots, x_{i_k,t_k})$. Therefore

$$\bar{a}(ij) = \begin{cases} 0, & \text{if} \quad i = i_r \quad \text{and} \quad j = t_r \quad \text{for some} \quad r \in [k], \\ \infty, & \text{otherwise.} \end{cases}$$

By our definition we have $\bar{a} = \beta(a)$, where $a \in A$ with

$$a(i) = \begin{cases} t_r - 1, & \text{if} \quad i = i_r \in \{i_1, \ldots, i_k\}, \\ \infty, & \text{otherwise.} \end{cases}$$

It will be enough to show that $a \in F$. Since $\bar{a} \in F_i^p$ and the restriction of β to F_i is a bijection from F_i to F_i^p, it follows that $a \in F_i$. If $a \notin F$, then there exists some $j \neq i$, such that $a \leq m_j$, and $\operatorname{infpt}(a) < \operatorname{infpt}(m_j)$. Therefore there exists an element $b \in F_j$, such that $b(i) = a(i)$ for all i with $b(i) \neq \infty$. This implies that $a < b$, and $\operatorname{infpt}(a) < \operatorname{infpt}(b) = \operatorname{infpt}(m_j)$. It follows from the definition of the map β that $\bar{a} < \bar{b}$, and that $P_{\bar{b}}$ is a prime ideal with $I^p \subset P_{\bar{b}} \subsetneqq P_{\bar{a}}$, a contradiction. $\square$

Now let $I \subset S$ be a monomial ideal and $I^p \subset T$ be the polarization of I. Furthermore let

$$\pi : T \longrightarrow S, \quad x_{i,j} \mapsto x_i.$$

be the epimorphism which attached to the polarization. Note that

$$\ker(\pi) = (x_{1,1} - x_{1,2}, \ldots, x_{1,1} - x_{1,r_1}, \ldots, x_{n,1} - x_{n,2}, \ldots, x_{n,1} - x_{n,r_n})$$

where r_i is the number of variables of the form $x_{i,j}$ which are needed for polarization. Set

$$y := x_{1,1} - x_{1,2}, \ldots, x_{1,1} - x_{1,r_1}, \ldots, x_{n,1} - x_{n,2}, \ldots, x_{n,1} - x_{n,r_n},$$

then y is a sequence of linear forms in T.

Proposition 2.2.9. *Let $I \subset S$ be a monomial ideal and I^p be the polarization of I. Assume that T/I^p is clean. Then there exists a clean filtration*

$$\mathcal{G}: \ I^p = J_0 \subset J_1 \subset \cdots \subset J_r = T$$

of I^p such that

$$\mathcal{F}: \ I = I_0 \subset I_1 \subset \cdots \subset I_r = S$$

the specialization of $\mathcal{G}$, that is, $\pi(J_i) = I_i$ for all i, is a pretty clean filtration of I with $I_k/I_{k-1} \cong S/\pi(Q_k)$, where $T/Q_k \cong J_k/J_{k-1}$.

Proof. For each $k \in [r]$ the S-module I_k/I_{k-1} is a cyclic module since J_k/J_{k-1} is cyclic for all k. Let $I_k/I_{k-1} \cong S/L_k$, where L_k is a monomial ideal in S. It is clear that $\pi(Q_k) \subset L_k$. Indeed, $Q_k = J_{k-1} : u_k$, where $J_k = (J_{k-1}, u_k)$ and where $J_k/J_{k-1} \cong T/Q_k$. If $v \in Q_k$, then $vu_k \in J_{k-1}$. It follows that $\pi(vu_k) = \pi(v)\pi(u_k) \in \pi(J_{k-1}) = I_{k-1}$, and hence $\pi(v) \in I_{k-1} : \pi(u_k) = L_k$.

We want to show that $\pi(Q_k) = L_k$. S and T are standard graded with $\deg(x_i) = \deg(x_{i,j}) = 1$ for all i and j, and $\mathcal{G}$ is a graded prime filtration of I^p. Therefore $\mathcal{F}$ is a graded filtration of I, and we have the following isomorphisms of graded modules $J_i/J_{i-1} \cong T/Q_i(-a_i)$ and $I_i/I_{i-1} \cong S/L_i(-a_i)$, where $a_i = \deg(u_i) = \deg(\pi(u_i))$.

The filtrations $\mathcal{G}$ and $\mathcal{F}$ yield the following Hilbert series of T/I^P and S/I:

$$\operatorname{Hilb}(T/I^p) = \sum_{i=1}^{r} \operatorname{Hilb}(T/Q_i)t^{a_i} \quad \text{and} \quad \operatorname{Hilb}(S/I) = \sum_{i=1}^{r} \operatorname{Hilb}(S/L_i)t^{a_i}.$$

Since y is a regular sequence of linear forms on T/I^p and on T/Q_i for each $i \in [r]$, we have

$$\begin{aligned} \operatorname{Hilb}(S/I) &= (1-t)^l \operatorname{Hilb}(T/I^p) = (1-t)^l \sum_{i=1}^{r} \operatorname{Hilb}(T/Q_i)t^{a_i} \\ &= \sum_{i=1}^{r} (1-t)^l \operatorname{Hilb}(T/Q_i)t^{a_i} = \sum_{i=1}^{r} \operatorname{Hilb}(S/\pi(Q_i))t^{a_i}, \end{aligned}$$

where $l = |y|$.

On the other hand, since $\pi(Q_i) \subset L_i$, we have the coefficientwise inequality $\operatorname{Hilb}(S/L_i) \leq \operatorname{Hilb}(S/\pi(Q_i))$, in other words, $\dim_K(S/L_i)_j \leq \dim_K(S/\pi(Q_i))_j$ for all j, and equality holds if and only if $L_i = \pi(Q_i)$. Therefore we have

$$\operatorname{Hilb}(S/I) = \sum_{i=1}^{r} \operatorname{Hilb}(S/\pi(Q_i))t^{a_i} \geq \sum_{i=1}^{r} \operatorname{Hilb}(S/L_i)t^{a_i} = \operatorname{Hilb}(S/I).$$

It follows that $L_i = \pi(Q_i)$ is a prime ideal for $i = 1, \ldots, r$.

We know that Γ^p the multicomplex associated to I^p is shellable, since T/I^p is clean. Therefore we may assume that $\mathcal{G}$ is obtained from a shelling of Γ^p. Also by [29, Corollary 10.7] and its proof we may assume that $\mu(Q_i) \geq \mu(Q_{i-1})$ for all $i \in [r]$, where $\mu(Q_i)$ is the number of generators of Q_i. Since by Corollary 2.2.6 each Q_i is of the form $(x_{i_1,t_1}, \ldots, x_{i_k,t_k})$ with $i_r \neq i_s$ for $r \neq s$, it follows that $\mu(Q_i) = \mu(\pi(Q_i)) = \mu(L_i)$. Therefore $\mu(L_i) \geq \mu(L_{i-1})$ for all i. This implies that $\mathcal{F}$ is a pretty clean filtration of S/I. $\square$

As the main result of this section we have

Theorem 2.2.10. *Let $I \subset S = K[x_1, \ldots, x_n]$ be a monomial ideal and I^p its polarization. Then the following are equivalent*

(a) *I is pretty clean.*

(b) *I^p is clean.*

Proof. (a) $\Rightarrow$ (b): Assume I is pretty clean. Then the multicomplex Γ associated with I is shellable. Let $a_1, \ldots, a_r$ be a shelling of Γ, and

$$\mathcal{F}: \ I = I_0 \subset I_1 \subset \ldots \subset I_r = S$$

the pretty clean filtration of I which is obtain from this shelling, i.e, $I_i = \bigcap\limits_{k=1}^{r-i} I(\Gamma(a_k))$. Let $\mathcal{F}^p$ be the polarization of $\mathcal{F}$. By Proposition 2.2.4, $\mathcal{F}^p$ is a prime filtration of I^p with $\ell(\mathcal{F}) = \ell(\mathcal{F}^p)$. Using Proposition 2.2.8 we have

$$|F(\Gamma^p)| = |F(\Gamma)|.$$

On the other hand, since I is pretty clean we know that $\ell(\mathcal{F}) = |F(\Gamma)|$. Hence we conclude that

$$|F(\Gamma^p)| = \ell(\mathcal{F}^p).$$

Therefore, since $\mathrm{Min}(I^p) = \mathrm{Ass}(I^p) \subset \mathrm{Supp}(\mathcal{F}^p)$, it follows that $\mathrm{Min}(I^p) = \mathrm{Supp}(\mathcal{F}^p)$, which implies that I^p is clean.

(b) $\Rightarrow$ (a): This follows from Proposition 2.2.9. $\square$

Corollary 2.2.11. *If $I \subset S$ is a monomial ideal which has no embedded prime ideal, then I is clean if and only if I^p is clean.* $\square$

As an other consequence we have the following:

Corollary 2.2.12. *Let $I \subset S$ be a monomial ideal. Then the following are equivalent:*

(a) *S/I is pretty clean;*

(b) *There exists a prime filtration*

$$\mathcal{F}: \ I = I_0 \subset I_1 \subset \cdots \subset I_r = S \quad \textit{where} \quad I_j/I_{j-1} \cong S/P_j$$

of S/I such that $\mu(P_i) \geq \mu(P_{i+1})$ for all $i = 0, \ldots, r-1$. $\square$

2.3 Cohen–Macaulay monomial ideals of codimension 2

In this section we show that all Cohen–Macaulay monomial ideals of codimension 2 are clean. To end this we need the following result.

Proposition 2.3.1. *Let $I \subset S$ be a monomial complete intersection ideal. Then S/I is clean.*

Proof. Let $G(I) = \{u_1, \ldots, u_m\}$ be the unique minimal set of monomial generators of I. By our assumption, $u_1, \ldots, u_m$ is a regular sequence. This implies that $\mathrm{supp}(u_i) \cap \mathrm{supp}(u_j) = \emptyset$ for all $i \neq j$.

It follows from the definition of the polarization of a monomial ideal (see 2.2), that for the polarized ideal $I^p = (u_1^p., \ldots, u_m^p)$ one again has $\mathrm{supp}(u_i^p) \cap \mathrm{supp}(u_j^p) = \emptyset$ for all $i \neq j$.

Thus $J = I^p \subset T$ is a squarefree monomial ideal generated by the regular sequence of monomials $v_1, \ldots, v_m$ with $v_i = u_i^p$ for all i.

Let Δ be the simplicial complex on vertex set $[t]$, (t is the number of variables in T) whose Stanley-Reisner ideal I_Δ is equal to J. The *Alexander dual* $\Delta^\vee$ of Δ is defined to be the simplicial complex whose faces are $\{[t] \setminus F\colon\ F \not\in \Delta\}$. The Stanley-Reisner ideal of $\Delta^\vee$ is minimally generated by all monomials $x_{i_1} \cdots x_{i_k}$ where $(x_{i_1}, \ldots, x_{i_k})$ is a minimal prime ideal of I_Δ.

In our case it follows that $I_{\Delta^\vee}$ is minimally generated by the monomials of the form $x_{i_1} \ldots x_{i_m}$ where $x_{i_j} \in \mathrm{supp}(v_j)$ for $j = 1, \ldots, m$. Thus we see that $I_{\Delta^\vee}$ is the matroidal ideal of the transversal matroid attached to the sets $\mathrm{supp}(v_1), \ldots, \mathrm{supp}(v_m)$, see [11, Section 5]. In [32, Lemma 1.3] and [11, Section 5] it is shown that any polymatroidal ideal has linear quotients, and this implies that Δ is a shellable simplicial complex, see for example [28, Theorem 1.4]. Hence by the theorem of Dress 1.4.4, T/I_Δ is clean. Hence by Theorem 2.2.10 we conclude that S/I is pretty clean. Since all prime ideals in a pretty clean filtration of S/I are associated prime ideals of S/I (see [29, Corollary 3.4]) and since S/I is Cohen-Macaulay, the prime ideals in the filtration are minimal. Hence S/I is clean. $\square$

Corollary 2.3.2. *Let $I \subset S$ be a monomial ideal with* $\mathrm{depth}\, S/I \geq n-1$. *Then S/I is pretty clean.*

Proof. The assumption implies that I is a principal ideal. Thus the assertion follows from Proposition 2.3.1. $\square$

With the same techniques as in the proof of Proposition 2.3.1 we can show the following:

Proposition 2.3.3. *Let $I \subset S$ be a monomial ideal which is perfect and of codimension 2. Then S/I is clean.*

Proof. We will show that the polarized ideal I^p defines a shellable simplicial complex. Then, as in the proof of Proposition 2.3.1, it follows that S/I is clean. Note that I^p is a perfect squarefree monomial ideal of codimension 2. Let Δ be the simplicial complex defined by I^p. By the Eagon–Reiner theorem [15] and a result of Terai [52], the ideal $I_{\Delta^\vee}$ has a 2-linear resolution. Now we use the fact, proved in [27, Theorem 3.2], that an ideal with 2-linear resolution has linear quotients which in turn implies that Δ is shellable, as desired. $\square$

Combining the preceding results with Lemma 2.1.4 we conclude the following:

Corollary 2.3.4. *Let $I \subset S$ be a monomial ideal. If $n \leq 4$ and S/I is Cohen-Macaulay, then S/I is clean.*

2.4 Gorenstein monomial ideals of codimension 3

As the main result of this section we will show

Theorem 2.4.1. *Each Gorenstein monomial ideal of codimension 3 is clean.*

The proof of this result is based on the following structure theorem that can be found in [8].

Theorem 2.4.2. *Let $I \subset S$ be a monomial Gorenstein ideal of codimension 3. Then $|G(I)|$ is an odd number, say $|G(I)| = 2m+1$, and there exists a regular sequence of monomials $u_1, \ldots u_{2m+1}$ in S such that*

$$G(I) = \{u_i u_{i+1} \cdots u_{i+m-1}\colon\ i = 1, \ldots, 2m+1\},$$

where $u_i = u_{i-2m-1}$ whenever $i > 2m+1$.

First we need to show

Proposition 2.4.3. *Let $I \subset T = K[y_1, \ldots, y_r]$ be a monomial ideal such that T/I is (pretty) clean. Let $u_1, \ldots, u_r \in S = K[x_1, \ldots, x_n]$ be a regular sequence of monomials, and let $\varphi\colon\ T \to S$ be the K-algebra homomorphism with $\varphi(y_j) = u_j$ for $j = 1, \ldots, r$. Then $S/\varphi(I)S$ is (pretty) clean.*

Proof. Let $I = I_0 \subset I_1 \subset \cdots \subset I_m = T$ be a pretty clean filtration $\mathcal{F}$ of T/I with $I_k/I_{k-1} = T/P_k$ for all k.

Observe that the K-algebra homomorphism $\varphi\colon T \to S$ is flat, since $u_1, \ldots, u_r$ is a regular sequence. Hence if we set $J_k = \varphi(I_k)S$ for $k = 1, \ldots, m$, then we obtain the filtration $\varphi(I)S = J_0 \subset J_1 \subset \cdots \subset J_m = S$ with $J_k/J_{k-1} \cong S/\varphi(P_k)S$.

Suppose $P_k = (y_{i_1}, \ldots, y_{i_k})$, then $\varphi(P_k)S = (u_{i_1}, \ldots, u_{i_k})$. In other words, $\varphi(P_k)S$ is a monomial complete intersection, and hence by Proposition 2.3.1 we have that $S/\varphi(P_k)S$ is clean. Therefore there exists a prime filtration $J_k = J_{k_0} \subset J_{k_1} \subset \cdots \subset J_{k_{r_k}} = J_{k+1}$ such that $J_{k_i}/J_{k_{i-1}} \cong S/P_{k_i}$ where P_{k_i} is a minimal prime ideal of $\varphi(P_k)S$. Since $\varphi(P_k)S = (u_{i_1}, \ldots, u_{i_{t_k}})S$ is a complete intersection, all minimal prime ideals of $\varphi(P_k)$ have height t_k.

Composing the prime filtrations of the J_k/J_{k-1}, we obtain a prime filtration of $S/\varphi(I)S$. We claim that this prime filtration is (pretty) clean. In fact, let P_{k_i} and P_{ℓ_j} be two prime ideals in the support of this filtration. We have to show: if $P_{k_i} \subset P_{\ell_j}$ for $k < \ell$, or $P_{k_i} \subset P_{\ell_j}$ for $k = \ell$ and $i < j$, then $P_{k_i} = P_{\ell_j}$. In case $k = \ell$, we have $\text{height}(P_{k_i}) = \text{height}(P_{\ell_j}) = t_k$, and the assertion follows. In case $k < \ell$, by using the fact that $\mathcal{F}$ is a pretty clean filtration, we have that $P_k = P_\ell$ or $P_k \not\subset P_\ell$. In the first case, the prime ideals P_{k_i} and P_{ℓ_j} have the same height, and the assertion follows. In the second case there exists a variable $y_g \in P_k \setminus P_\ell$. Then the monomial u_g belongs to $\varphi(P_k)S$ but not to $\varphi(P_\ell)S$. This implies that P_{k_i} contains a variable which belongs to the support of u_g. However this variable cannot be a generator of P_{ℓ_j}, because the support of u_g is disjoint of the support of all the monomial generators of $\varphi(P_\ell)S$. This shows that $P_{k_i} \not\subset P_{\ell_j}$. □

Corollary 2.4.4. *Let Δ be a shellable simplicial complex and $I_\Delta \subset T = K[y_1, \ldots, y_r]$ its Stanley-Reisner ideal. Furthermore, let $u_1, \ldots, u_r \subset S = K[x_1, \ldots, x_n]$ be a regular sequence of monomials, and let $\varphi(y_i) = u_i$ for $i = 1, \ldots, r$. Then $S/\varphi(I_\Delta)S$ is clean.*

Proof. By Theorem 1.4.4 the ring T/I_Δ is clean. Therefore, $S/\varphi(I_\Delta)S$ is again clean, by Proposition 2.4.3. □

Proof of Theorem 2.4.1. Let Δ be the simplicial complex whose Stanley-Reisner ideal

$$I_\Delta \subset T = K[y_1, \ldots, y_{2m+1}]$$

is generated by the monomials $y_i y_{i+1} \cdots y_{i+m-1}$, $i = 1, \ldots, 2m+1$, where $y_i = y_{i-2m-1}$ whenever $i > 2m + 1$, and let $u_1, \ldots, u_{2m+1} \subset S = K[x_1, \ldots, x_n]$ be the regular sequence given in Theorem 2.4.1. Then we have $I = \varphi(I_\Delta)S$ where $\varphi(y_j) = u_j$ for all j. Therefore, by Corollary 2.4.4, it suffices to show that Δ is shellable.

Identifying the vertex set of Δ with $[2m + 1] = \{1, \ldots, 2m + 1\}$ and observing that I_Δ is of codimension 3, it is easy to see that $F \subset [2m+1]$ is a facet of Δ if and only if $F = [2m + 1] \setminus \{a_1, a_2, a_3\}$ with

$$a_2 - a_1 < m + 1, \qquad a_3 - a_2 < m + 1, \qquad a_3 - a_1 > m.$$

We denote the facet $[2m + 1] \setminus \{a_1, a_2, a_3\}$ by $\mathrm{F}(a_1, a_2, a_3)$

We will show that Δ is shellable with respect to the lexicographic order. Note that $\mathrm{F}(a_1, a_2, a_3) < \mathrm{F}(b_1, b_2, b_3)$ in the lexicographic order, if and only if either $b_1 < a_1$, or $b_1 = a_1$ and $b_2 < a_2$, or $a_1 = b_1$, $a_2 = b_2$ and $b_3 < a_3$.

In order to prove that Δ is shellable we have to show: if $F = \mathrm{F}(a_1, a_2, a_3)$ and $G = \mathrm{F}(b_1, b_2, b_3)$ with $F < G$, then there exists $c \in G \setminus F$ and some facet H such that $H < G$ and $G \setminus H = \{c\}$.

We know that $|G \setminus F| \leq 3$. If $|G \setminus F| = 1$, then there is nothing to prove. In the following we discuss the cases $|G \setminus F| = 2$ and $|G \setminus F| = 3$. The discussion of these cases is somewhat tedious but elementary. For the convenience of the reader we list all the possible cases.

Case 1: $|G \setminus F| = 2$.

(i) If $b_1 = a_1 < b_2 < a_2$, then we choose $H = (G \setminus \{a_2\}) \cup \{b_2\}$.

(ii) If $b_1 < b_2 = a_1$ or $b_1 < b_2 < a_1 < a_2 = b_3 < a_3$, then we choose $H = (G \setminus \{a_3\}) \cup \{b_1\}$.

(iii) If $b_1 < a_1 < b_2 < a_2 = b_3 < a_3$, we consider the following two subcases:

for $a_3 - b_2 < m+1$, we choose $H = (G \setminus \{a_3\}) \cup \{b_3\}$.

for $a_3 - b_2 \geq m+1$, we choose $H = (G \setminus \{a_3\}) \cup \{b_1\}$.

(iv) If $b_1 < a_1 < a_2 = b_2 < b_3 < a_3$, then we choose $H = (G \setminus \{a_3\}) \cup \{b_3\}$.

(v) If $b_1 < a_1 < a_2 = b_2 < a_3 < b_3$ or $b_1 < a_1 < a_2 < a_3 = b_2 < b_3$, then we choose $H = (G \setminus \{a_1\}) \cup \{b_1\}$.

Case 2: $|G \setminus F| = 3$.

(i) If $b_1 < a_1 < a_2 < a_3 < b_3$, then we choose $H = (G \setminus \{a_1\}) \cup \{b_1\}$.

(ii) If $b_1 < b_2 < b_3 < a_1 < a_2 < a_3$ or $b_1 < b_2 < a_1 < a_2 < a_3$ and $a_1 < b_3$, then we choose $H = (G \setminus \{a_1\}) \cup \{b_2\}$.

(iii) If $b_1 < a_1 < b_2 < b_3 < a_2 < a_3$, then we choose $H = (G \setminus \{a_2\}) \cup \{b_3\}$.

(iv) If $b_1 < a_1 < b_2 < a_2 < b_3 < a_3$, we consider the following two subcases:

for $a_3 - b_2 < m+1$, we choose $H = (G \setminus \{a_3\}) \cup \{b_3\}$.

for $a_3 - b_2 \geq m+1$, we choose $H = (G \setminus \{a_3\}) \cup \{b_1\}$.

(iv) If $b_1 < a_1 < a_2 < b_2 < b_3 < a_3$, then we choose $H = (G \setminus \{a_3\}) \cup \{b_3\}$. □

Combining the result of Theorem 2.4.1 with Corollary 2.3.2, Proposition 2.3.3 and the result of Apel [6, Corollary 3] we obtain

Corollary 2.4.5. *Let $I \subset S$ be monomial ideal. If $n \leq 5$ and S/I is Gorenstein, then S/I clean.*

2.5 Monomial ideals of forest type

Let Δ be a simplicial complex with $\mathcal{F}(\Delta) = \{F_1, \ldots, F_t\}$. Recall from preliminaries that a facet F_i is called a *leaf* of Δ if F_i is the only facet of Δ, or there exists a facet F_j, $j \neq i$ such that $F_i \cap F_k \subseteq F_i \cap F_j$ for any $k \neq i$. The facet F_j is called a *branch* of F_i. A simplicial complex Δ is called a forest if any subcomplex Γ of Δ, i.e $\mathcal{F}(\Gamma) \subset \mathcal{F}(\Delta)$, has a leaf. It is easy to see that if F_i is a leaf of Δ and F_j a branch of F_i, then $\gcd(x_{F_i}, x_{F_k}) | \gcd(x_{F_i}, x_{F_j})$ for any $k \neq i$, where $x_F = \prod_{i \in F} x_i$.

In this section we study a class of monomial ideals which are pretty clean. This class is a generalization of the class of facet ideals of forests.

Let $I \subset S$ be a squarefree monomial ideal. There is a unique simplicial complex Δ such that $I = I(\Delta)$. Now we generalize the definition of the facet ideal of a forest to any monomial ideal. Let I be a monomial ideal with $G(I) = \{u_1, \ldots, u_m\}$. A variable x_i is called a free variable of I if there exists a $t \in [m]$ such that $x_i \mid u_t$ and $x_i \nmid u_j$ for any $j \neq t$. A monomial u_t is called a *leaf* of $G(I)$ if u_t is the only generator of I, or there exists a $j \in [m]$, $j \neq t$ such that $\gcd(u_t, u_i) \mid \gcd(u_t, u_j)$ for all $i \neq t$. In this case u_j is called a *branch* of u_t. We say that I is a *monomial ideal of forest type* if any subset of $G(I)$ has a leaf.

Notice that if u, v are two monomials, then

$$\gcd(u,v)^p = \gcd(u^p, v^p),$$

where u^p denote the polarization of u. The following follows immediately from the definitions.

Lemma 2.5.1. *Let $I = (u_1, \ldots, u_m)$ be a monomial ideal. Then I is a monomial ideal of forest type if and only if $I^p = (u_1^p, \ldots, u_m^p)$ is a facet ideal of some forest Δ.*

Then it is clear that if I is a monomial ideal of forest type, then I^p has a free variable.

Let $(X_1, X_2) = (\{x_{i_1}, \ldots, x_{i_r}\}, \{x_{j_1}, \ldots, x_{j_s}\})$, where X_1, X_2 are subsets of $X = \{x_1, \ldots, x_n\}$ and $X_1 \cap X_2 = \emptyset$. Let I be a monomial ideal in $S = K[x_1, \ldots, x_n]$. As in [53] we define the *minor* of I with respect to (X_1, X_2) to be the ideal $I_{(X_1,X_2)} \subset K[X \setminus X_1 \cup X_2]$ obtained from I by setting $x_{i_k} = 0$ and $x_{j_l} = 1$ for $k = 1, \ldots, r$ and $l = 1, \ldots, s$. In particular, $I_{(\emptyset,\emptyset)} = I$. One says that the ideal I has the *free variable property* if all minors of I have free variables. The following lemma is a generalization of [17, Lemma 4.5] to any monomial ideal of forest type.

Lemma 2.5.2. *Let I be a monomial ideal of forest type and $X' = \{x_{j_1}, \ldots, x_{j_s}\}$ a subset of X. Then $I_{(\emptyset,X')}$ is again a monomial ideal of forest type.*

Proof. We only need to prove that $I_{(\emptyset,\{x_{j_1}\})}$ is a monomial ideal of forest type. Hence we may assume that $X' = \{x_i\}$. Let $G(I) = \{u_1, \ldots, u_m\}$. We write $u_j = \bar{u}_j x_i^{a_j}$, where $a_j \geq 0$ and $x_i \nmid \bar{u}_j$ for $j = 1, \ldots, m$. Let A be any subset of $G(I_{(\emptyset,X')})$. Consider the subset $A' = \{u_j \colon \bar{u}_j \in A\}$ of $G(I)$. Since I is a monomial ideal of forest type, A' has a leaf u_p. This means that there exists a $u_k \in A'$ such that $\gcd(u_p, u_q) \mid \gcd(u_p, u_k)$ for all $u_q \in A'$ with $q \neq p$.

Let $\gcd(u_p, u_q) = v_q x_i^{a_q}$ and $\gcd(u_p, u_k) = v_k x_i^{a_k}$, where v_q, v_k are monomials and $x_i \nmid v_q$, $x_i \nmid v_k$. Then $\gcd(\bar{u}_p, \bar{u}_q) = v_q$ which divides $\gcd(\bar{u}_p, \bar{u}_k) = v_k$ for all $\bar{u}_q \in A$ with $q \neq p$. Hence $\bar{u}_p$ is a leaf of A. $\square$

Now we recall the following fact from [38].

Lemma 2.5.3. *Let $K \subset S$ be a monomial ideal and u a monomial in S which is regular over S/K. Then S/K is pretty clean if and only if $S/(K, u)$ is pretty clean.*

The following proposition is crucial for proving one of the main results of this section.

Proposition 2.5.4. *Let $I \subset S$ be a monomial ideal with $G(I) = \{u_1, \ldots, u_{m-1}, \bar{u}_m x_j^t\}$ where x_j is a free variable of I. If $I_{(\emptyset,\{x_j\})}$ and $I_{(\{x_j\},\emptyset)}$ are pretty clean, then I is pretty clean.*

Proof. We denote $I_{(\emptyset,\{x_j\})} = (u_1,\dots,u_{m-1},\bar{u}_m)$ and $I_{(\{x_j\},\emptyset)} = (u_1,\dots,u_{m-1})$ by J and K respectively. It is easy to see that $J/I = (I,\bar{u}_m)/I \cong S/(I:\bar{u}_m) = S/(K,x_j^t)$. Since S/K is pretty clean, by Lemma 2.5.3 J/I is also pretty clean. Let $\mathcal{F}_1\colon I = I_0 \subset I_1 \subset \cdots \subset I_r = J$ be a pretty clean filtration of J/I with $I_i/I_{i-1} \cong S/P_i$. Then by [29, Corollary 3.4] $\mathrm{Supp}(\mathcal{F}_1) = \mathrm{Ass}(J/I) = \mathrm{Ass}(S/(K,x_j^t))$. Hence $x_j \in P_i$ for $i = 1,\dots,r$.

By our assumption S/J is pretty clean. Let $\mathcal{F}_2\colon J = I_r \subset I_{r+1} \subset \cdots \subset I_{r+s} = S$ be a pretty clean filtration of S/J with $I_{r+i}/I_{r+i-1} \cong S/P_{r+i}$. Then $P_{r+i} \in \mathrm{Ass}(S/J)$. Hence $x_j \notin P_{r+i}$ for $i = 1,\dots,s$.

Combining the prime filtrations $\mathcal{F}_1$ and $\mathcal{F}_2$ we get the prime filtration

$$\mathcal{F}\colon I = I_0 \subset \cdots \subset I_r = J \subset I_{r+1} \subset \cdots \subset I_{r+s} = S$$

of S/I. Since $x_j \in P_i$ for $i = 1,\dots,r$ and $x_j \notin P_{r+i}$ for $i = 1,\dots,s$, one has $P_i \nsubseteq P_{r+t}$ for any $i \in [r]$ and any $t \in [s]$. Therefore $\mathcal{F}$ is a pretty clean filtration of S/I since $\mathcal{F}_1$ and $\mathcal{F}_2$ are pretty clean filtrations. □

Combining Proposition 2.5.4 with Lemma 2.5.2, we get the following theorem.

Theorem 2.5.5. *If $I \subset S$ is a monomial ideal of forest type, then S/I is pretty clean.*

Proof. Let $I^p \subset T$ be the polarization of I. Then by Lemma 2.5.1 I^p is a monomial ideal of forest type. By Theorem 2.2.10 S/I is pretty clean if and only if T/I^p is clean. Therefore we may assume that I is squarefree. We use induction on n the number of variables to prove the assertion. Let $G(I) = \{u_1,\dots,u_m\}$ and let x_i be a free vertex of I. We may assume that $u_m = \bar{u}_m x_i^a$ with $a > 0$. By Lemma 2.5.2, the ideal $J = (u_1,\dots,u_{m-1},\bar{u}_m)$ is a monomial ideal of forest type. It is clear that $K = (u_1,\dots,u_{m-1})$ is also a monomial ideal of forest type. By induction hypothesis S/J and S/K are pretty clean. Therefore by Proposition 2.5.4, S/I is pretty clean. □

It follows from [29, Corollary 4.3] that if S/I is pretty clean, then S/I is sequentially Cohen–Macaulay. Therefore we have the following corollary, which generalizes the main result of Faridi [18].

Corollary 2.5.6. *If $I \subset S$ is a monomial ideal of forest type, then S/I is sequentially Cohen–Macaulay.*

The notion of *good leaf* was introduced in the thesis of Zheng [58]. A leaf F of a simplicial complex Δ is called a good leaf if F is a leaf of each subcomplex Γ of Δ to which F belongs. Equivalently, F is a good leaf of Δ if the collection of sets $F \cap G$ with $G \in \mathcal{F}(\Delta)$ is totally ordered with respect to inclusion. Let Δ be a simplicial complex and $I(\Delta)$ its facet ideal. We say that x_F is a good leaf of $I(\Delta)$ if F is a good leaf of Δ. An order $F_1,\dots,F_t$ of the facets of Δ is called a good leaf order if F_i is a good leaf of the subcomplex $\langle F_1,\dots,F_i\rangle$ for $i = 1,\dots,t$. It is obvious that if $F_1, \ldots, F_t$ is a good leaf order, then $\Delta = \langle F_1, \ldots, F_t \rangle$ is a forest. It was shown in [26, Corollary 3.4] that any forest has a good leaf order. Therefore a simplicial complex Δ is a forest if and only if it has a good leaf order. The notion of good leaf and good leaf order naturally can be extended to any monomial ideal.

Let I be a monomial ideal with the minimal set of generators $G(I)$. A leaf u of $G(I)$ is a good leaf if u is a leaf of each subset of $G(I)$ to which u belongs. Equivalently, u is a good leaf of $G(I)$ if the collection of monomials $\gcd(u, v)$ with $v \in G(I)$ is totally ordered with respect to divisibility. An order $u_1, \ldots, u_m$ of monomials in $G(I)$ is a good leaf order of $G(I)$ if u_i is a good leaf of $G(I_i)$ for $i = 1, \ldots, m$, here $I_i = (u_1, \ldots, u_i)$.

Lemma 2.5.7. *Let $I = (u_1, \ldots, u_m)$ be a monomial ideal and $I^p = (u_1^p, \ldots, u_m^p)$ its polarization. Then $u_1, \ldots, u_m$ is a good leaf order of I if and only if $u_1^p, \ldots, u_m^p$ is a good leaf order of I^p.*

As a consequence of Lemma 2.5.1, Lemma 2.5.7 and [26, Corollary 3.4] we get the following:

Corollary 2.5.8. *Any monomial ideal of forest type has a good leaf order.*

It is easy to see that if $u_1, \ldots, u_m$ is a good leaf order of $G(I)$, then I is a monomial ideal of forest type. Using Corollary 2.5.8 it follows immediately that a monomial ideal I is of forest type if and only if $G(I)$ admits a good leaf order.

A sequence of monomials $u_1, \ldots, u_r$ in a set of indeterminates $X = \{x_1, \ldots, x_n\}$ is said to be an M-sequence if for all $1 \leq i < j \leq r$ there exists a new numbering of the variables that appear in u_i such that if $u_i = x_{i_1}^{a_1} \cdots x_{i_s}^{a_s}$ and $x_{i_k} | u_j$ for some $k \leq s$, then $x_{i_k}^{a_k} \cdots x_{i_s}^{a_s} | u_j$. Notice that the new numbering of variables may depend on the index i. This notion was introduced in [10]. It is easy to see that if $u_1, \ldots, u_u$ is an M-sequence, then any subsequence of it in the same order is again an M-sequence.

Proposition 2.5.9. *Let $I = (u_1, \ldots, u_m)$ be a monomial ideal. If $u_1, \ldots, u_m$ is an M-sequence, then $u_m, \ldots, u_1$ is a good leaf order of I and hence I is a monomial ideal of forest type.*

Proof. It is enough to show that u_1 is a good leaf of I. Since $u_1, \ldots, u_m$ is an M-sequence, there is a new numbering of variables which appear in u_1 such that if $u_1 = x_1^{a_1} x_2^{a_2} \cdots x_t^{a_t}$, then $\gcd(u_1, u_i) = x_{j_i}^{a_{j_i}} \cdots x_t^{a_t}$ for all $i = 2, \ldots, m$. Therefore the set of monomials $\gcd(u_1, u_i)$, $i = 2, \ldots, m$ is totally ordered with respect to divisibility. This implies that u_1 is a good leaf of I. $\square$

The following example shows that the converse of Proposition 2.5.9 is not true in general.

Example 2.5.10. Let $I = (x^2yz^3, x^3y^2z)$. Then I is a monomial ideal of forest type and x^2yz^3, x^3y^2z is a good leaf order. But I is not generated by an M-sequence.

However it was shown in [58, Proposition 3.11] that if $F_1, \ldots, F_t$ is a good leaf order of a forest Δ, then $x_{F_t}, \ldots, x_{F_1}$ is an M-sequence. If we combine [58, Proposition 3.11] with Proposition 2.5.9 we get

Theorem 2.5.11. *Let I be a squarefree monomial ideal. Then I is generated by an M-sequence if and only if I is the facet ideal of a forest Δ.*

Now we introduce a class of monomial ideals. Then we will show that this class is contained in the class of monomial ideals of forest type. Let $\mathcal{I}$ be the class of monomial ideals with the following properties:

(a) any irreducible monomial ideal is in $\mathcal{I}$;

(b) if $I \in \mathcal{I}$, then I has a free variable;

(c) if x_i is a free variable of I, then $I \in \mathcal{I}$ if and only if the minors $I_{(\emptyset,\{x_i\})}$ and $I_{(\{x_i\},\emptyset)}$ are in $\mathcal{I}$.

It is obvious that if a monomial ideal I has free variable property, then $I \in \mathcal{I}$. Moreover we have the following:

Theorem 2.5.12. *Let $I \subset S$ be a monomial ideal. Then we have*

(i) *I has free variable property if and only if $I \in \mathcal{I}$.*

(ii) *If $I \in \mathcal{I}$, then I is a monomial ideal of forest type;*

Proof. (i): This is obvious.

(ii): We show that I is a monomial ideal of forest type by using induction on the number of variables n which appear in I. The case $n = 1$ is clear. Let $n > 1$. Since $I \in \mathcal{I}$, we may assume that $G(I) = \{u_1, \ldots, u_m\}$, where $u_m = \bar{u}_m x_t^a$ and x_t is a free variable of I. Since the ideals $J = (u_1, \ldots, u_{m-1}, \bar{u}_m)$ and $K = (u_1, \ldots, u_{m-1})$ are in $\mathcal{I}$ with less variables, by induction hypothesis J and K are monomial ideals of forest type. Let A be any subset of $G(I)$. If $u_m \notin A$, then $A \subset G(K)$. Hence it has a leaf. If $u_m \in A$ and $\bar{u}_m \mid u_j$ for some $u_j \in A$ and $j \neq m$, then $\gcd(u_m, u_j) = \bar{u}_m$ and $\gcd(u_m, u_i) \mid \bar{u}_m$ for any $i \neq m$. This means that u_m is a leaf of A. Now we may assume that $u_m \in A$ and $\bar{u}_m \nmid u_j$ for any $u_j \in A$ and $j \neq m$. Then $A' = (A \setminus \{u_m\}) \cup \{\bar{u}_m\}$ is a subset of $G(J)$ and hence it has a leaf. Let u_p be a leaf of A'. Since x_t is a free variable, we have $\gcd(u_m, u_i) = \gcd(\bar{u}_m, u_i)$ for any $i \neq m$. If $u_p = \bar{u}_m$, then u_m is a leaf of A. If $u_p \neq \bar{u}_m$, then u_p itself is a leaf of A. $\square$

Remark 2.5.13. If I is a squarefree monomial ideal, then the following are equivalent

(i) I is a monomial ideal of forest type;

(ii) I has free variable property ;

(iii) $I \in \mathcal{I}$.

A *clutter* $\mathcal{C}$ with vertex set $[n]$ is a family of subsets of $[n]$, called *edges*, with the property that non of them is contained in another. The edge ideal of a clutter $\mathcal{C}$ is defined to be the ideal $I(\mathcal{C}) = (x_C \colon C \text{ is an edge of } \mathcal{C})$, where $x_C = \prod_{i\in C} x_i$. A clutter is a special kind of hypergraph. A hypergraph H on the vertex set $[n]$ is a family of subsets of $[n]$. One may also view a clutter $\mathcal{C}$ as the set of facets of some simplicial complex Δ. In this case, $I(\mathcal{C}) = I(\Delta)$.

In [53], the authors say a clutter $\mathcal{C}$ has free vertex property if the edge ideal $I(\mathcal{C})$ has free variable property. By Theorem 2.5.12 one sees that $\mathcal{C}$ has free vertex property if and only if $I(\mathcal{C})$ is a monomial ideal of forest type. If we consider $\mathcal{C}$ to be the set of facets of some simplicial complex Δ, then $\mathcal{C}$ has free vertex property if and only if Δ is a forest. In the following we denote by $\Delta_{\mathcal{C}}$ the simplicial complex whose Stanley–Reisner ideal is $I(\mathcal{C})$.

As a corollary of Theorem 2.5.12 and Theorem 2.5.5, we obtain the following:

Corollary 2.5.14. *([53, Theorem 5.3]) If the clutter $\mathcal{C}$ has the free vertex property, then $S/I(\mathcal{C})$ is clean, i.e. $\Delta_{\mathcal{C}}$ is shellable.*

Let $\mathcal{C}$ be a clutter and Δ the simplcial complex such that $I(\mathcal{C}) = I(\Delta)$. We say that the clutter $\mathcal{C}$ is a forest if Δ is a simplcial forest. Up to the order of the vertices and the order of the edges, a clutter is determined by its incidence matrix and vice versa. The incidence matrix $M_{\mathcal{C}}$ is defined as follows: let $1, \ldots, n$ be the vertices and $C_1, \ldots, C_m$ be the edges of the clutter $\mathcal{C}$. Then $M_{\mathcal{C}} = (e_{ij})$ is an $n \times m$ matrix with $e_{ij} = 1$ if $i \in C_j$ and $e_{ij} = 0$ if $i \notin C_j$. A clutter is called *totally balanced* if its incidence matrix has no square submatrix of order at least 3 with exactly two 1's in each row and column. It is known that a totally balanced clutter has free vertex property, see [42, Corollary 83.3a]. On the other hand, in [26, Theorem 3.2], it is shown that $\mathcal{C}$ is a forest if and only if $\mathcal{C}$ is totally balanced. These together with Theorem 2.5.11 and Theorem 2.5.12 imply the following:

Corollary 2.5.15. *Let $\mathcal{C}$ be a clutter. The following statements are equivalent:*

(i) *$I(\mathcal{C})$ is generated by an M-sequence;*

(ii) *$\mathcal{C}$ is a forest;*

(iii) *$\mathcal{C}$ is totally balanced;*

(iv) *$\mathcal{C}$ has free vertex property.*

2.6 A new characterization of pretty clean monomial ideals

Let R be a Noetherian ring, and M a finitely generated R-module. For $P \in \operatorname{Spec}(R)$ the number $\operatorname{mult}_M(P) = \ell(H^0_P(M_P))$ is called the *length multiplicity* of P with respect to M where $H^0_P(M_P)$ is the 0-th local cohomology of M_P. Obviously, one has $\operatorname{mult}_M(P) > 0$ if and only if $P \in \operatorname{Ass}(M)$. Assume now that $(R, \mathfrak{m})$ is a local ring. Recall that the *arithmetic degree* of M is defined to be

$$\operatorname{adeg}(M) = \sum_{P \in \operatorname{Ass}(M)} \operatorname{mult}_M(P) \cdot \operatorname{e}(R/P),$$

where $\operatorname{e}(R/P)$ is the *multiplicity* of the associated graded ring of R/P.

First we notice the following

Lemma 2.6.1. *Suppose R is a Noetherian ring, and M a finitely generated R-module. Let*

$$\mathcal{F}\colon 0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

be a prime filtration of M with $M_i/M_{i-1} \cong R/P_i$. Then

$$\operatorname{mult}_M(P) \leq |\{i \in [r-1] : \; M_{i+1}/M_i \cong R/P\}|$$

for all $P \in \operatorname{Spec}(R)$.

Proof. If $P \notin \operatorname{Ass}(M)$, the assertion is trivial. So now let $P \in \operatorname{Ass}(M)$. Localizing at P we may assume that P is the maximal ideal of M.

Now we will prove the assertion by induction on $\ell(\mathcal{F})$. If $\ell(\mathcal{F})$=1, then the assertion is obviously true. Let $\ell(\mathcal{F}) > 1$. From the following short exact sequence

$$0 \to M_1 \to M \to M/M_1 \to 0$$

we get the following long exact sequence

$$0 \to H^0_P(M_1) \to H^0_P(M) \to H^0_P(M/M_1) \to \dots$$

Therefore $\operatorname{mult}_M(P) = \ell(H^0_P(M)) \leq \ell(H^0_P(M_1)) + \ell(H^0_P(M/M_1))$. By induction hypothesis

$$\operatorname{mult}_{M/M_1}(P) = \ell(H^0_P(M/M_1)) \leq |\{i \in [r-1] \backslash \{1\} : \; M_{i+1}/M_i \cong R/P\}|.$$

Now consider the following two cases:

(i) If $M_1 \cong R/P$, then $\ell(H^0_P(M_1)) = 1$. Therefore

$$\operatorname{mult}_M(P) \leq 1 + \operatorname{mult}_{M/M_1}(P) \leq |\{i \in [r-1] : \; M_{i+1}/M_i \cong R/P\}|.$$

(ii) If $M_1 \ncong R/P$, then $\ell(H^0_P(M_1)) = 0$. Hence

$$\operatorname{mult}_M(P) \leq |\{i \in [r-1] : \; M_{i+1}/M_i \cong R/P\}|. \qquad \square$$

Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over the field K. Let $I \subset S$ be a monomial ideal and Γ be the multicomplex associated to I. We denote the arithmetic degree of S/I by $\operatorname{adeg}(I)$. Since $\operatorname{e}(S/P) = 1$ for all $P \in \operatorname{Ass}(I)$, it follows that $\operatorname{adeg}(I) = \sum_{P \in \operatorname{Ass}(I)} \operatorname{mult}_I(P)$, where $\operatorname{mult}_I(P) = \operatorname{mult}_{S/I}(P)$. By [50, Lemma 3.3] $\operatorname{adeg}(I) = |\operatorname{Std}(I)|$, where $\operatorname{Std}(I)$ is the set of standard pairs with respect to I. Also by [29, Lemma 9.14] $|\operatorname{Std}(I)| = |F(\Gamma)|$. Since $|F(\Gamma)| = |F(\Gamma^p)|$, see Proposition 2.2.8, it follows that $\operatorname{adeg}(I) = \operatorname{adeg}(I^p)$, where I^p is the polarization of I and Γ^p the multicomplex associated to I^p.

In this part we want to show that $\operatorname{adeg}(I)$ is a lower bound for the length of any prime filtration of S/I and the equality holds if and only if S/I is a pretty clean module.

Lemma 2.6.2. *Let $I \subset S$ be a monomial ideal and $\mathcal{F}$ a prime filtration of I. One has*

(i) $\operatorname{adeg}(I) \leq \ell(\mathcal{F})$;

(ii) $\ell(\mathcal{F}) = \operatorname{adeg}(I) \Leftrightarrow \mathcal{F}$ *is a pretty clean filtration of I.*

Proof. Part (i) is clear by Lemma 2.6.1.

One direction of (ii) is [29, Corollary 6.4]. For the other direction assume $\ell(\mathcal{F}) = \operatorname{adeg}(I) = |F(\Gamma)| = |F(\Gamma^p)|$. By Proposition 2.2.4 $\mathcal{F}^p$ is a prime filtration of I^p with $\ell(\mathcal{F}^p) = |F(\Gamma^p)|$= the number of minimal prime ideals of Γ^p. Therefore $\mathcal{F}^p$ is a clean filtration of I^p, so by Theorem 2.2.10 $\mathcal{F}$ is a pretty clean filtration of I. □

Combining Lemma 2.6.2 with Theorem 2.2.10 we get

Theorem 2.6.3. *Let $I \subset S$ be a monomial ideal. Assume Γ is the multicomplex associated to I and I^p the polarization of I. The following are equivalent:*

(a) Γ *is shellable;*

(b) I *is pretty clean;*

(c) *There exists a prime filtration $\mathcal{F}$ of I with $\ell(\mathcal{F}) = \operatorname{adeg}(I)$;*

(d) I^p *is clean;*

(e) *If $\triangle$ be the simplicial complex associated to I^p, then $\triangle$ is shellable.*

If R is a Noetherian ring and M a finitely generated R-module with pretty clean filtration $\mathcal{F}$, then $\operatorname{Ass}(M) = \operatorname{Supp}(\mathcal{F})$, see [29, Corollary 3.6]. The converse is not true in general as shown in [29, Example 4.4]. The example given there is a cyclic module defined by a non-monomial ideal. The following example shows that even in the monomial case the converse does not hold in general.

Example 2.6.4. Let $S = K[a,b,c,d]$ be the polynomial ring over the field K, $I \subset S$ the ideal

$$I = (a,b)\cdot(c,d)\cdot(a,c,d) = (abc, abd, acd, ad^2, a^2d, ac^2, a^2c, bcd, bc^2, bd^2)$$

and $M = S/I$. We claim that the module $M = S/I$ is not pretty clean, but that M has a prime filtration $\mathcal{F}$ with $\mathrm{Supp}(\mathcal{F}) = \mathrm{Ass}(M)$.

Note that $(a,b)\cap(c,d)\cap(a,c,d^2)\cap(a,c^2,d)\cap(a^2,b,c,d^2)\cap(a^2,b,c^2,d)$ modulo I is an irredundant primary decomposition of (0) in M.

We see that $\mathrm{Ass}(M) = \{(a,b),(c,d),(a,c,d),(a,b,c,d)\}$. It is clear that

$$\begin{aligned}\mathcal{F}:\ I &= I_0 \subset I_1 = (I,ac) \subset I_2 = (I_1,ad) \subset I_3 = (I_2,bd)\\ &\subset I_4 = (I_3,bc) \subset I_5 = (I_4,a) \subset I_6 = (a,b) \subset S\end{aligned}$$

is a prime filtration of M with $\mathrm{Supp}(\mathcal{F}) = \mathrm{Ass}(M)$. Indeed $I_1/I \cong I_2/I_1 \cong S/(a,b,c,d)$, $I_3/I_2 \cong I_4/I_3 \cong I_6/I_5 \cong S/(a,c,d)$ and $I_5/I_4 \cong S/(c,d)$.

From the above irredundant primary decomposition of I it follows that $\mathrm{adeg}(I)$=6. But the length of any prime filtration of I is at least 7. Therefore I can not be pretty clean. In other words, from [29, Corollary 1.2] it follows that $D_1(M) = ((a,b)\cap(c,d))/I$ and that $D_2(M) = M$, where $D_i(M)$ is the largest submodule of M with $\dim(M) \le i$, for $i = 0,\ldots,\dim(M)$. It follows that $D_2(M)/D_1(M) \cong S/(a,b)\cap(c,d)$ is not clean. Knowing now $D_2(M)/D_1(M)$ is not clean, we conclude from [29, Corollary 4.2] that $M = S/I$ is not pretty clean.

3 Stanley decompositions and partitions

In this chapter we study Stanley decompositions of $\mathbb{Z}^n$-graded S-modules. In [46, Conjecture 5.1] Stanley conjectured the following: let R be a finitely generated $\mathbb{N}^n$-graded K-algebra (where $R_0 = K$ as usual), and let M be a finitely generated $\mathbb{Z}^n$-graded R-module. Then there exist finitely many subalgebras $S_1, \ldots, S_t$ of R, each generated by algebraically independent $\mathbb{N}^n$-homogeneous elements of R, and there exist $\mathbb{Z}^n$-homogeneous elements $m_1, \ldots, m_t$ of M, such that

$$M = \bigoplus_{i=1}^{t} m_i S_i$$

where $\dim S_i \geq \operatorname{depth} M$ for all i, and where $m_i S_i$ is a free S_i-module (of rank one). Moreover, if K is infinite and under a given specialization to an $\mathbb{N}$-grading R is generated by R_1, then we can choose the ($\mathbb{N}^n$-homogeneous) generators of each S_i to lie in R_1.

Stanley's conjecture has been studied in several articles, see for examples [5], [6], [44], [31], [3], [4], [37] and [51].

We consider this conjecture in the case that M is a finitely generated $\mathbb{Z}^n$-graded S-module, where $S = K[x_1, \ldots, x_n]$ is the polynomial ring in n variables. It is known that the associated prime ideals of M are monomial ideals, and any monomial prime ideal is of the form $P_F = (x_i : i \in F)$ for some $F \subset [n]$. Let $m \in M$ be a homogeneous element and $Z \subset \{x_1, \ldots, x_n\} = X$. We denote by $mK[Z]$ the K-subspace of M generated by all homogeneous elements of the form mu, where u is a monomial in $K[Z]$. The K-subspace $mK[Z]$ is called a *Stanley space of dimension* $|Z|$ if $mu \neq 0$ for any non-zero monomial $u \in K[Z]$.

A decomposition $\mathcal{D}$ of M as a finite direct sum of Stanley spaces is called a *Stanley decomposition* of M. The minimal dimension of a Stanley space in the decomposition $\mathcal{D}$ is called the *Stanley depth* of $\mathcal{D}$, denoted $\operatorname{sdepth}(\mathcal{D})$. We set

$$\operatorname{sdepth}(M) = \max\{\operatorname{sdepth}(\mathcal{D}) \colon \mathcal{D} \text{ is a Stanley decomposition of } M\},$$

and call this number the *Stanley depth* of M. With the above notation Stanley's conjecture says that $\operatorname{depth}(M) \leq \operatorname{sdepth}(M)$.

Apel [6] showed that if $I \subset S$ is a monomial ideal, then

$$\operatorname{sdepth}(S/I) \leq \min\{\dim(S/P) \colon P \in \operatorname{Ass}(S/I)\}.$$

The same result is true for any finitely generated $\mathbb{Z}^n$-graded S-module M. Indeed, let

$$\mathcal{D} = \bigoplus_{i=1}^{t} m_i K[Z_i]$$

be a Stanley decomposition of M such that $\operatorname{sdepth}(\mathcal{D}) = \operatorname{sdepth}(M)$ and $P \in \operatorname{Ass}(M)$ an associated prime such that $\dim(S/P) = \min\{\dim(S/Q) \colon Q \in \operatorname{Ass}(M)\}$.

Since $P \in \operatorname{Ass}(M)$, there exists a non-zero homogeneous element $m \in M$ such that $P = \operatorname{Ann}(m)$. On the other hand since $0 \neq m \in M$, there exists a unique $1 \leq k \leq t$ such that $m \in m_k K[Z_k]$. It is enough to show that $Z_k \cap P = \emptyset$. Let $m = m_k x_F$ for some $x_F \in K[Z_k]$. Suppose that $Z_k \cap P \neq \emptyset$, and choose $x_i \in Z_k \cap P$. Then $m_k(x_F x_i) = m x_i = 0$, a contradiction. This implies that $|Z_k| \leq \dim(S/P)$. In particular,

$$\operatorname{sdepth}(M) = \operatorname{sdepth}(\mathcal{D}) \leq \dim(S/P) = \min\{\dim(S/Q)\colon Q \in \operatorname{Ass}(M)\}.$$

Let R be a finitely generated standard graded K-algebra where K is a field, and let M be a finitely generated graded R-module. The Hilbert series of M is defined to be $\operatorname{Hilb}(M) = \sum\limits_{i \in Z} (\dim_K M_i) t^i$. It is known that if $\dim(M) = d$, then there exists a $Q_M(t) \in Z[t, t^{-1}]$ such that

$$\operatorname{Hilb}(M) = Q_M(t)/(1-t)^d$$

and $Q_M(1) \neq 0$. The number $Q_M(1)$ is called the multiplicity of M, and is denoted by $\mathrm{e}(M)$.

Let $I \subset S$ be a monomial ideal. The number of Stanley spaces of a given dimension in a Stanley decomposition may depend on this particular decomposition. For example, if $I = (xy) \subset K[x, y]$, then for all integers $k > 0$ and $l > 0$ we have the Stanley decomposition

$$S/I = x^l K[x] \oplus y^k K[y] \oplus \left(\bigoplus_{i=0}^{l-1} x^i K\right) \oplus \left(\bigoplus_{j=1}^{k-1} y^j K\right),$$

for S/I with as many Stanley spaces of dimension 0 as we want, however only 2 Stanley spaces of dimension 1 in any Stanley decomposition. This is a general fact. Indeed we have the following:

Proposition 3.0.5. *Let M be a $\mathbb{Z}^n$-graded S-module of dimension d. Then the number of Stanley spaces of maximal dimension d is independent of the special Stanley decomposition of M. In fact, this number is equal to the multiplicity, $e(M)$, of M.*

Proof. Let

$$M = \bigoplus_{i=1}^{r} m_i K[Z_i]$$

be an arbitrary Stanley decomposition of M, and $d = \max\{|Z_i|\colon i = 1, \ldots, r\}$. Then

$$\operatorname{Hilb}(M) = \sum_{i=1}^{r} \operatorname{Hilb}(m_i K[Z_i]) = \sum_{i=1}^{r} t^{\deg(m_i)}/(1-t)^{|Z_i|} = Q_M(t)/(1-t)^d.$$

with $Q_M(t) = \sum\limits_{i=1}^{r} (1-t)^{d-|Z_i|} t^{\deg(m_i)}$. It follows that $e(M) = Q_M(1)$ is equal to the number of Stanley space of dimension d in this Stanley decomposition of M. $\square$

Corollary 3.0.6. *Let $I \subset S$ be a monomial ideal such that S/I is Cohen-Macaulay. Then the following conditions are equivalent:*

(a) *Stanley's conjecture is true for S/I.*

(b) *There exists a Stanley decomposition $\mathcal{D}$ of S/I such that each Stanley space in $\mathcal{D}$ has dimension $d = \dim S/I$.*

(c) *There exists a Stanley decomposition $\mathcal{D}$ of S/I which has $e(S/I)$ summands.*

3.1 Prime filtrations and Stanley decompositions

Let $I \subset S$ be a monomial ideal and $I = I_0 \subset I_1 \subset \cdots \subset I_r = S$ an $\mathbb{N}^n$-graded prime filtration of S/I with $I_i/I_{i-1} \cong S/P_{F_i}(-\mathbf{a}_i)$ where $F_i \subset [n]$ and $P_{F_i} = (x_j : j \in F_i)$. It was shown in [29, page 398] that this prime filtration of S/I give us the Stanley decomposition

$$S/I = \bigoplus_{i=1}^{r} u_i K[Z_{F_i^c}]$$

of S/I, where $Z_{F_i^c} = \{x_j : j \notin F_i\}$, and where $u_i = x^{\mathbf{a}_i}$. This Stanley decomposition is called the Stanley decomposition of S/I corresponding to the given prime filtration. In general we have the following:

Proposition 3.1.1. *Let M be a finitely generated $\mathbb{Z}^n$-graded S-module. If $(0) = M_0 \subset M_1 \subset \cdots \subset M_r = M$ is a prime filtration of M such that $M_i/M_{i-1} \cong S/P_{F_i}(-\mathbf{a}_i)$, then*

$$M \cong \bigoplus_{i=1}^{r} m_i K[Z_{F_i^c}]$$

is a Stanley decomposition of M where $m_i \in M_i$ is a homogeneous element of degree $\mathbf{a}_i$ such that

$$(M_{i-1} :_S m_i) = P_{F_i} \quad \text{and} \quad Z_{F_i^c} = \{x_j : j \notin F_i\}.$$

Proof. We prove the assertion by induction on r the length of prime filtration. If $r = 1$, then $M \cong S/P_{F_1}(-\mathbf{a_1})$. Therefore M is cyclic module and there exists a homogeneous generator $m_1 \in M$ of degree $\mathbf{a_1}$ such that $0 :_S m_1 = P_{F_1}$. Hence $M \cong m_1 K[Z_{F_1^c}]$. Now let $r > 1$. Then the $\mathbb{Z}^n$-graded S-module M/M_1 has a prime filtration

$$(0) = M_1/M_1 \subset M_2/M_1 \subset \cdots \subset M_r/M_1 = M/M_1$$

which has length $r - 1$. By induction hypothesis $M/M_1 \cong \bigoplus_{i=2}^{r} m_i K[Z_{F_i^c}]$. On the other hand from the short exact sequence $0 \to M_1 \to M \to M/M_1 \to 0$ we have $M = M_1 \oplus M/M_1$ as a graded K-vector space. Since $M_1 \cong m_1 K[Z_{F_1^c}]$, one has $M \cong \bigoplus_{i=1}^{r} m_i K[Z_{F_i^c}]$. $\square$

As an immediate consequence of Proposition 3.1.1 and proposition 3.0.5 we have the following:

Corollary 3.1.2. *Let M be a $\mathbb{Z}^n$-graded S-module of dimension d. The number of prime ideals of height $n-d$ which appear in any prime filtration of M is independent of the special prime filtration of M. In fact, this number is equal to $e(M)$ the multiplicity of M.*

Let $I \subset S$ be a monomial ideal. We also note that for each monomial $u \in \tilde{I} \setminus I$ the 0-dimensional Stanley space uK belongs to any Stanley decomposition of S/I. In fact $u\mathfrak{m}^k \subset I$ for some k. Now if u belongs to some Stanley space $vK[Z]$ with $|Z| \geq 1$, then $vK[Z] \cap I \neq \emptyset$, a contradiction.

Apel [6] studied some cases in which Stanley's conjecture holds for S/I. Theorem 6.5 in [29] proves that for all pretty clean monomial ideals Stanley's conjecture holds. Therefore combining Theorem 2.1.7, Lemma 2.1.4, Proposition 2.3.1, Proposition 2.3.3, Corollary 2.3.4, Theorem 2.4.1, Corollary 2.4.5 and Theorem 2.5.5 with [29, Theorem 6.5] we get

Theorem 3.1.3. *Let $I \subset S = K[x_1, \dots, x_n]$ be a monomial ideal. Then in the following cases Stanley's conjecture holds for S/I.*

(a) *If I is a monomial ideal of height $\geq n-1$;*

(b) ([6, Theorem 4]) *If $n \leq 3$;*

(c) *If I is a complete intersection monomial ideal,* (This follows also from [6, Therorem 3])*;*

(d) *If I is a perfect monomial ideal of codimension 2;*

(e) *If I is Cohen–Macaulay and $n \leq 4$;*

(f) *If I is a Gorenstein monomial ideal of codimension 3;*

(g) *If I is Gorenstein monomial ideal and $n \leq 5$;*

(h) *If I is monomial ideal of forest type.*

In the proof of [29, Theorem 6.5] it is used that Stanley decompositions of S/I arise from a pretty clean filtration of S/I. Recall that if $\mathcal{F}$ is a pretty clean filtration of M, then $\mathrm{Ass}(M) = \mathrm{Supp}(\mathcal{F})$. The converse of this statement is not always true, see Example 2.6.4. As a generalization of [29, Theorem 6.5] we show

Proposition 3.1.4. *Suppose M is a $\mathbb{Z}^n$-graded S-module, and $\mathcal{F}$ is a prime filtration of M with $\mathrm{Supp}(\mathcal{F}) = \mathrm{Ass}(M)$. Then the Stanley decomposition of M which is obtained from this prime filtration satisfies the condition of Stanley's conjecture.*

Proof. The Stanley decomposition which is obtained from $\mathcal{F}$ has the property that $|Z_i| = \dim S/P_i$, see Proposition 3.1.1. By [7, Proposition 1.2.13] we have $\operatorname{depth}(M) \leq \dim(S/P_i)$ for all $P_i \in \operatorname{Ass}(M)$, and hence the assertion follows. $\square$

In all cases discussed above we found a Stanley decomposition corresponding to a prime filtration and satisfying the Stanley conjecture. However we will show that there exist examples of monomial ideals such that *all* Stanley decompositions arising from a prime filtration may fail to satisfy the Stanley conjecture.

Remark 3.1.5. Let $I \subset S$ be a Cohen-Macaulay monomial ideal, and

$$\mathcal{F}: \; I = I_0 \subset I_1 \subset \cdots \subset I_r = S$$

be a prime filtration of S/I. We claim that if the Stanley decomposition of S/I corresponding to $\mathcal{F}$ satisfies the Stanley conjecture, then $\operatorname{Ass}(I) = \operatorname{Supp}(\mathcal{F})$. In particular I is clean, since $\operatorname{Min}(I) = \operatorname{Ass}(I)$.

Indeed, since I is Cohen-Macaulay we have $\operatorname{depth}(S/I) = \dim(S/I) = \dim(S/P)$ for all $P \in \operatorname{Ass}(I)$. We recall that $I_i/I_{i-1} \cong S/P_i(-\mathbf{a}_i)$ for suitable $\mathbf{a}_i \in \mathbb{N}^n$ and that $P_i \in \operatorname{Ass}(I_{i-1})$ for $i = 1, \ldots, r$. Let $T_i = u_i K[Z_i]$ be the Stanley space corresponding to $S/P_i(-\mathbf{a}_i)$ as explained as above. Then $|Z_i| = \dim(S/P_i)$. Assume that $P_i \notin \operatorname{Ass}(I)$ for some $i > 1$. Since $I \subset I_{i-1} \subset P_i$, there exists a $P_j \in \operatorname{Ass}(I)$ such that $P_j \subsetneq P_i$. It follows that $|Z_i| = \dim(S/P_i) < \dim(S/P_j) = \operatorname{depth}(S/I)$, a contradiction.

Example 3.1.6. Let K be a field and

$$I = (abd, abf, ace, adc, aef, bde, bcf, bce, cdf, def) \subset S = K[a, b, c, d, e, f].$$

The ideal I is the Stanley-Reisner ideal corresponding to the simplicial complex Δ which is the triangulation of the real projective plane $\mathbb{P}^2$, see [7, Figure 5.8]. It is known that S/I is Cohen-Macaulay if and only if $\operatorname{char}(K) \neq 2$. This implies S/I is not clean, since otherwise Δ would be shellable and S/I would be Cohen-Macaulay for any field K. Hence by Remark 3.1.5, if $\operatorname{char}(K) \neq 2$, no Stanley decomposition of S/I which corresponds to a prime filtration of S/I satisfies the Stanley conjecture. Nevertheless S/I has the following Stanley decomposition which satisfies Stanley's conjecture and hence does not come from a prime filtration.

$$\begin{aligned} S/I &= K[c, f, e] \oplus dK[d, c, e] \oplus bK[b, d, c] \oplus aK[a, d, e] \oplus abK[a, b, c] \oplus afK[a, f, d] \\ &\oplus acK[a, c, f] \oplus bfK[b, f, e] \oplus beK[a, b, e] \oplus dfK[b, d, f]. \end{aligned}$$

Unfortunately not all Stanley decompositions of S/I correspond to prime filtrations, even if S/I is pretty clean. Such an example is given by Mclagan and Smith in [36]. Let $I = (x_1x_2x_3) \subset K[x_1, x_2, x_3]$. Then

$$S/I = 1 \oplus x_1K[x_1, x_2] \oplus x_2K[x_2, x_3] \oplus x_3K[x_1, x_3]$$

is a Stanley decomposition of S/I which does not correspond to a prime filtration of S/I. On the other hand, by Theorem 2.1.7 we know that S/I is pretty clean.

Now we want to characterize those Stanley decompositions of S/I which correspond to a prime filtration of S/I.

Lemma 3.1.7. *Let $I \subset S = K[x_1, \ldots, x_n]$ be a monomial ideal, and $T = uK[Z]$ be a Stanley space in a Stanley decomposition of S/I. The K-vector space $I_1 = I \oplus T$ is a monomial ideal if and only if $I_1 = (I, u)$. In this case, $I : u = P$, where $P = (x_i \colon x_i \notin Z)$.*

Proof. We have $I \subset I_1$ and $u \in I_1$. Suppose now that I_1 is a monomial ideal. Since (I, u) is the smallest monomial ideal that contains I and u, it follows that $(I, u) \subset I_1$. On the other hand, $I_1 = I + uK[Z] \subset I + uK[x_1, \ldots, x_n] = (I, u)$. Hence $I_1 = (I, u)$.

Since for each $x_i \notin Z$ we have $x_i u \in I_1 = I \oplus T$ and $x_i u \notin uK[Z] = T$, it follows that $x_i u \in I$ and hence $x_i \in I : u$. On the other hand, if $v \in K[Z]$ is a monomial, then $vu \notin I$, since $uK[Z]$ is a Stanley space of S/I. Therefore $I : u = P = (x_i \colon x_i \notin Z)$. □

Corollary 3.1.8. *The monomial ideal $I \subset S$ is a prime ideal if and only if there exists a Stanley decomposition of S/I consisting of only one Stanley space.* □

As a consequence of this Lemma we have

Proposition 3.1.9. *Let $I \subset S$ be a monomial ideal, and $S/I = \bigoplus_{i=1}^{r} u_i K[Z_i]$ be a Stanley decomposition of S/I. The given Stanley decomposition corresponds to a prime filtration of S/I if and only if the Stanley spaces $T_i = u_i K[Z_i]$ can be ordered $T_1, \ldots, T_r$, such that*

$$I_k = I \oplus T_1 \oplus \ldots \oplus T_k$$

is a monomial ideal for $k = 1, \ldots, r$.

Proof. We prove " if " by induction on r. If $r = 0$ then the assertion is trivially true. Let $r \geq 1$. By assumption $I_1 = I \oplus T_1$ is a monomial ideal. Hence by Lemma 3.1.7 we have $I_1 = (I, u_1)$ and $I : u_1 = P_1 = (x_i \colon x_i \notin Z_1)$. We notice that in this case $I_1/I \cong S/P_1(-\mathbf{a}_1)$ and $u_1 = \prod_{j=1}^{n} x_j^{\mathbf{a}_1(j)}$, and that $S/I_1 = \bigoplus_{i=2}^{r} T_i$. Now by the induction hypothesis this Stanley decomposition of S/I_1 corresponds to a prime filtration, say $\mathcal{F}_1$

$$\mathcal{F}_1 : \ I_1 \subset I_2 \subset \cdots \subset I_r = S.$$

Therefore the given Stanley decomposition of S/I corresponds to the prime filtration

$$\mathcal{F} : \ I \subset I_1 \subset I_2 \subset \cdots \subset I_r = S.$$

The converse follows immediately if we order the Stanley spaces of S/I which are obtained from a prime filtration according to the order of the ideals in this filtration. □

The following definition is due to Maclagan and Smith [36].

Definition 3.1.10. Let $I \subset S$ be a monomial ideal. A *Stanley filtration* of S/I is a Stanley decomposition of S/I with an ordering of the Stanley spaces $\{u_iK[Z_i] : 1 \leq i \leq r\}$ such that for all $1 \leq j \leq r$ the set $\{u_iK[Z_i] : 1 \leq i \leq j\}$ is a Stanley decomposition of S/I_j where $I_j = I + (u_{j+1}, \dots, u_r)$. Equivalently, the ordered set $\{u_iK[Z_i] : 1 \leq i \leq r\}$ is a Stanley filtration provided the modules S/I_j form a filtration $K = S/I_0 \subsetneq S/I_1 \subsetneq \cdots \subsetneq S/I_r = S/I$ with $\frac{S/I_j}{S/I_{j-1}} \cong K[x_i : i \notin Z_i]$.

Remark 3.1.11. Let $I \subset S$ be a monomial ideal and $\{u_iK[Z_i] : 1 \leq i \leq r\}$ be a Stanley decomposition of S/I. The given decomposition of S/I is a Stanley filtration if and only if is correspond to a prime filtration of S/I. Indeed if $\{u_iK[Z_i] : 1 \leq i \leq r\}$ is a Stanley filtration, then $\{u_iK[Z_i] : 1 \leq i \leq r-1\}$ is a Stanley decomposition of S/I_{r-1}, where $I_{r-1} = I+(u_r)$, i.e $\oplus_{i=1}^{r-1} u_iK[Z_i] \oplus I_{r-1}K[X] = S$. On the other hand $\oplus_{i=1}^{r-1} u_iK[Z_i] \oplus u_rK[Z_r] \oplus IK[X] = S$. Therefore $I + u_rK[Z_r] = (I, u_r)$ is a monomial ideal. Now by induction on r, one can order the given Stanley decomposition such that $I_{r-i} = I \oplus (\oplus_{j=r-i+1}^{r} u_jK[Z_j])$ is a monomial ideal for $i = 0, \dots, r$. Hence by Proposition 3.1.9 this decomposition is correspond to a prime filtration. The converse is easy, because if one order the Stanley spaces in the Stanley decomposition according to order of the ideals in the prime filtration.

We conclude this section by showing

Corollary 3.1.12. *If $I \subset S = K[x, y]$ is a monomial ideal, then each Stanley decomposition of S/I corresponds to a prime filtration of S/I.*

Proof. The K-vector space $\tilde{I}/I$ has finite dimension, say m. So we can choose monomials $v_1, \dots, v_m \in \tilde{I}$ whose residue classes modulo I form a K-basis for $\tilde{I}/I$. As observed in the discussions before Proposition 3.1.3, in any Stanley decomposition of S/I these monomials have to appear as 0-dimensional Stanley spaces. In the proof of Lemma 2.1.1 we showed that it is possible to order the monomials $v_1, \dots, v_m$ in such a way that

$$I_i = I \oplus v_1K \oplus \ldots \oplus v_iK = (I, v_1, \dots, v_i)$$

is a monomial ideal for $i = 1 \dots, m$. If we remove in the given Stanley decomposition of S/I the Stanley spaces v_iK, $i = 1, \dots, m$, the remaining summands establish a Stanley decomposition of $S/\tilde{I}$. Thus we may assume that I is saturated. Hence $I = (x^\alpha y^\beta)$.

Let $S/I = \bigoplus_{i=1}^{r} u_iK[Z_i]$ be a Stanley decomposition of S/I. We will prove by induction on $\alpha + \beta$ that the given Stanley decomposition can be ordered such that $I_k = I \oplus (\bigoplus_{i=1}^{k} u_iK[Z_i])$ is a monomial ideal for all k. If $\alpha + \beta = 0$ the assertion is trivially true. Let $\alpha + \beta > 0$. The Stanley decomposition of S/I contains at least one summand of the form $x^{\alpha-1}y^\gamma K[y]$, where $\gamma \geq \beta$, or $x^\theta y^{\beta-1}K[x]$, where $\theta \geq \alpha$.

We may assume that $x^{\alpha-1}y^{\gamma}K[y]$ is one of the summands. Let $t = \gamma - \beta$, and set $v_i = x^{\alpha-1}y^{\gamma-i+1}$ for $i = 1, \ldots, t+1$. If we set $T_1 = v_1K[y]$, then $I_1 = I \oplus T_1 = (I, v_1)$ is a monomial ideal. If we remove the Stanley space T_1 from the given Stanley decomposition of S/I, the remaining establish a Stanley decomposition of S/I_1. Since $v_2, \ldots, v_{t+1}$ belong to $\tilde{I}_1 \setminus I_1$, these monomials have to appear in any Stanley decomposition of S/I_1 as 0-dimensional Stanley spaces. In particular these monomials appear as 0-dimensional Stanley space, $T_2 = v_2K, \ldots, T_{t+1} = v_{t+1}K$ in the given Stanley decomposition of S/I. Now it is clear that $I_i = I_{i-1} \oplus T_i = (I_{i-1}, v_i)$ is a monomial ideal for $i = 1, \ldots, t+1$, where $I_0 = I$.

Removing the Stanley spaces $T_1, \ldots, T_{t+1}$ from the given Stanley decomposition of S/I, the remaining summands establish a Stanley decomposition of S/I_{t+1}. Since $I_{t+1} = (x^{\alpha-1}y^{\beta})$ is a saturated ideal, the assertion follows by the induction hypothesis applied to S/I_{t+1}. $\square$

3.2 Squarefree Stanley decompositions and partitions of simplicial complexes

Let $u \in S$ be a monomial and $Z \subset X = \{x_1, \ldots, x+n\}$. A Stanley space $uK[Z]$ is called a *squarefree Stanley space*, if u is a squarefree monomial and $\mathrm{supp}(u) \subseteq \mathrm{supp}(Z)$. We shall use the following notation: for $F \subseteq [n]$ we set $x_F = \prod_{i \in F} x_i$ and $Z_F = \{x_i\colon\ i \in F\}$. Then a Stanley space is squarefree if and only if it is of the form $x_FK[Z_G]$ with $F \subseteq G \subseteq [n]$.

A Stanley decomposition of S/I is called a *squarefree Stanley decomposition* of S/I, if all Stanley spaces in the decomposition are squarefree.

Let $I \subset S$ a monomial ideal. Denote by $I^c \subset S$ the K-linear subspace of S spanned by all monomials which do not belong to I. Then $S = I^c \oplus I$ as a K-vector space, and the residues of the monomials in I^c form a K-basis of S/I. Hence as a K-vector space we have $I^c \cong S/I$.

Note that I and I^c as well as all Stanley spaces are K-linear subspaces of S with a basis which is a subset of monomials of S. For any K-linear subspace $U \subset S$ which is generated by monomials, we denote by $\mathrm{Mon}(U)$ the set of elements in the monomial basis of U. It is then clear that if $u_iK[Z_i]$, $i = 1, \ldots, r$ are Stanley spaces, then $I^c = \bigoplus_{i=1}^r u_iK[Z_i]$ if and only if $\mathrm{Mon}(I^c)$ is the disjoint union of the sets $\mathrm{Mon}(u_iK[Z_i])$.

Lemma 3.2.1. *Let $I \subset S$ be a monomial ideal. The following conditions are equivalent:*

(a) *I is a squarefree monomial ideal.*

(b) *S/I has a squarefree Stanley decomposition.*

Proof. (a) $\Rightarrow$ (b): We may view I as the Stanley-Reisner ideal of some simplicial complex Δ. With each $F \in \Delta$ we associate the squarefree Stanley space $x_FK[Z_F]$.

We claim that $\bigoplus_{F\in\Delta} x_F K[Z_F]$ is a (squarefree) Stanley decomposition of S/I. Indeed, a monomial $u \in S$ belongs to I^c if and only if $\operatorname{supp}(u) \in \Delta$, and these monomial form a K-basis for I^c. On the other hand, a monomial $u \in S$ belongs to $x_F K[Z_F]$ if and only if $\operatorname{supp}(u) = F$. This shows that $I^c = \bigoplus_{F\in\Delta} x_F K[Z_F]$.

(b) $\Rightarrow$ (a): Let $\bigoplus_i u_i K[Z_i]$ be a squarefree Stanley decomposition of S/I. Assume that I is not a squarefree monomial ideal. Then there exists $u \in G(I)$ which is not squarefree and we may assume that $x_1^2 | u$. Then $u' = u/x_1 \in I^c$, and hence there exists i such that $u' \in u_i K[Z_i]$. Since $x_1 | u'$ it follows that $x_1 \in Z_i$. Therefore $u \in u_i K[Z_i] \subset I^c$, a contradiction. $\square$

Let Δ be a simplicial complex of dimension $d-1$ on the vertex set $[n] = \{1, \ldots, n\}$. A subset $\mathcal{I} \subset \Delta$ is called an *interval*, if there exits faces $F, G \in \Delta$ such that $\mathcal{I} = \{H \in \Delta\colon F \subseteq H \subseteq G\}$. We denote this interval given by F and G also by $[F, G]$ and call $\dim G - \dim F$ the *rank* of the interval. A *partition* $\mathcal{P}$ of Δ is a presentation of Δ as a disjoint union of intervals. The r-vector of $\mathcal{P}$ is the integer vector $r = (r_0, r_1, \ldots, r_d)$ where r_i is the number of intervals of rank i.

Proposition 3.2.2. *Let $\mathcal{P}\colon\ \Delta = \bigcup_{i=1}^r [F_i, G_i]$ be a partition of Δ. Then*

(a) *$D(\mathcal{P}) = \bigoplus_{i=1}^r x_{F_i} K[Z_{G_i}]$ is squarefree Stanley decomposition of S/I.*

(b) *The map $\mathcal{P} \mapsto D(\mathcal{P})$ establishes a bijection between partitions of Δ and squarefree Stanley decompositions of S/I.*

Proof. (a) Since each $x_{F_i} K[Z_{G_i}]$ is a squarefree Stanley space it suffices to show that I^c is indeed the direct sum of the Stanley spaces $x_{F_i} K[Z_{G_i}]$. Let $u \in \operatorname{Mon}(I^c)$; then $H = \operatorname{supp}(u) \in \Delta$. Since $\mathcal{P}$ is a partition of Δ it follows that $H \in [F_i, G_i]$ for some i. Therefore, $u = x_{F_i} u'$ for some monomial $u' \in K[Z_{G_i}]$. This implies that $u \in x_{F_i} K[Z_{G_i}]$. This shows that $\operatorname{Mon}(I^c)$ is the union of sets $\operatorname{Mon}(x_{F_i} K[Z_{G_i}])$. Suppose there exists a monomial $u \in x_{F_i} K[Z_{G_i}] \cap x_{F_j} K[Z_{G_j}]$. Then $\operatorname{supp}(u) \in [F_i, G_i] \cap [F_j, G_j]$. This is only possible if $i = j$, since $\mathcal{P}$ is partition of Δ.

(b) Let $[F_i, G_i]$ and $[F_j, G_j]$ be two intervals. Then $x_{F_i} K[Z_{G_i}] = x_{F_j} K[Z_{G_j}]$ if and only if $[F_i, G_i] = [F_j, G_j]$. Indeed, if $x_{F_i} K[Z_{G_i}] = x_{F_j} K[Z_{G_j}]$, then $x_{F_j} \in x_{F_i} K[Z_{G_i}]$, and hence $x_{F_i} | x_{F_j}$. By symmetry we also have $x_{F_j} | x_{F_i}$. In other words, $F_i = F_j$, and it also follows that $K[Z_{G_i}] = K[Z_{G_j}]$. This implies $G_i = G_j$. These considerations show that $\mathcal{P} \mapsto D(\mathcal{P})$ is injective.

On the other hand, let $\mathcal{D}\colon\ S/I = \bigoplus_{i=1}^r x_{F_i} K[Z_{G_i}]$ be an arbitrary squarefree Stanley decomposition of S/I. By the definition of a squarefree Stanley set we have $F_i \subseteq G_i$, and since $x_{F_i} K[Z_{G_i}] \subset I^c$, it follows that $G_i \in \Delta$. Hence $[F_i, G_i]$ is an interval of Δ, and a squarefree monomial x_F belongs to $x_{F_i} K[Z_{G_i}]$ if and only if $F \in [F_i, G_i]$.

Let $F \subset \Delta$ be an arbitrary face. Then $x_F \in \operatorname{Mon}(I^c) = \bigcup_{i=1}^r \operatorname{Mon}(x_{F_i} K[Z_{G_i}])$. Hence the squarefree monomial x_F belongs to $x_{F_i} K[Z_{G_i}]$ for some i, and hence $F \in [F_i, G_i]$. This shows that $\bigcup_{i=1}^r [F_i, G_i] = \Delta$. Suppose $F \in [F_i, G_i] \cap [F_j, G_j]$. Then $x_F \in x_{F_i} K[Z_{G_i}] \cap x_{F_j} K[Z_{G_j}]$, a contradiction. Hence we see that $\mathcal{P}\colon\ \Delta = \bigcup_{i=1}^r [F_i, G_i]$ is a partition of Δ with $D(\mathcal{P}) = \mathcal{D}$. $\square$

Now let $I \subset S$ be a squarefree monomial ideal. Then we set

$$\operatorname{sqdepth}(S/I) = \max\{\operatorname{sdepth}(\mathcal{D})\colon \mathcal{D} \text{ is a squarefree Stanley decomposition of } S/I\},$$

and call this number the *squarefree Stanley depth* of S/I.

As the main result of this section we have

Theorem 3.2.3. *Let $I \subset S$ be a squarefree monomial ideal. Then*

$$\operatorname{sqdepth}(S/I) = \operatorname{sdepth}(S/I).$$

Proof. Let $\mathcal{D}$ be any Stanley decomposition of S/I, and let Δ be the simplicial complex with $I = I_\Delta$. For each $F \in \Delta$ we have $x_F \in I^c$. Hence there exists a summand $uK[Z]$ with $x_F \in uK[Z]$. Since x_F is squarefree it follows that $u = x_G$ is squarefree and $F \subseteq G \cup Z$. Let $\mathcal{D}'$ the sum of those Stanley spaces $uK[Z]$ in $\mathcal{D}$ for which u is a squarefree monomial. Then this sum is direct. Therefore the intervals $[G, G \cup Z]$ corresponding to the summands in $\mathcal{D}'$ are pairwise disjoint. On the other hand these intervals cover Δ, as we have seen before, and hence form a partition of $\mathcal{P}$ of Δ. It follows from the construction of $\mathcal{P}$ that $\operatorname{sqdepth} D(\mathcal{P}) \geq \operatorname{sdepth} \mathcal{D}$. This shows that $\operatorname{sqdepth}(S/I) \geq \operatorname{sdepth}(S/I)$. The other inequality $\operatorname{sqdepth}(S/I) \leq \operatorname{sdepth}(S/I)$ is obvious. $\square$

Corollary 3.2.4. *Let Δ be a simplicial complex. Then the following conditions are equivalent:*

(a) *Stanley's conjecture holds for S/I_Δ.*

(b) *There exists a partition $\Delta = \bigcup_{i=1}^r [F_i, G_i]$ with $|G_i| \geq \operatorname{depth} S/I_\Delta$ for all i.*

Let Δ be a simplicial complex and $\mathcal{F}(\Delta)$ its set of facets. Stanley calls a simplicial complex Δ *partitionable* if there exists a partition $\Delta = \bigcup_{i=1}^r [F_i, G_i]$ with $\mathcal{F}(\Delta) = \{G_1, \ldots, G_r\}$. We call a partition with this property a *nice partition.* Stanley conjectures [47, Conjecture 2.7] (see also [48, Problem 6]) that each Cohen-Macaulay simplicial complex is partitionable. In view of Corollary 3.2.4 it follows that the conjecture of Stanley decompositions implies the conjecture on partitionable simplicial complexes. More precisely we have

Corollary 3.2.5. *Let Δ be a Cohen-Macaulay simplicial complex with h-vector $(h_0, h_1, \ldots, h_d)$. Then the following conditions are equivalent:*

(a) *Stanley's conjecture holds for S/I_Δ.*

(b) *Δ is partitionable.*

(c) *Δ admits a partition whose r-vector satisfies $r_i = h_{d-i}$ for $i = 0, \ldots, d$.*

(d) *Δ admits a partition into $e(S/I_\Delta)$ intervals.*

Moreover, any nice partition of Δ *satisfies the conditions* (c) *and* (d).

Proof. (a) $\Longleftrightarrow$ (b) follows from Corollary 3.2.4. In order to prove the implication (b) $\Rightarrow$ (c), consider a nice partition $\Delta = \bigcup_{i=1}^{r}[F_i, G_i]$ of Δ. From this decomposition the f-vector of Δ can be computed by the following formula

$$\sum_{i=0}^{d} f_{i-1}t^i = \sum_{i=0}^{d} r_i t^{d-i}(1+t)^i.$$

On the other hand one has

$$\sum_{i=0}^{d} f_{i-1}t^i = \sum_{i=0}^{d} h_i t^i(1+t)^{d-i},$$

see [7, p. 213]. Comparing coefficients the assertion follows.

The implication (c) $\Rightarrow$ (d) follows from the fact that $e(K[\Delta]) = \sum_{i=0}^{d} h_i$, see [7, Proposition 4.1.9]. Finally (d) $\Rightarrow$ (a) follows from Corollary 3.0.6. □

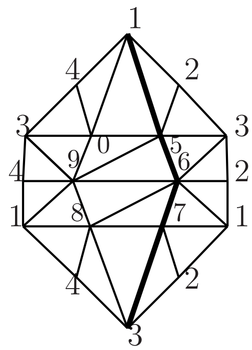
3.3 Some examples

We conclude this chapter with some explicit examples. Recall that constructibility, a generalization of shellability, is defined recursively as follows: (i) a simplex is constructible, (ii) if Δ_1 and Δ_2 are d-dimensional constructible complexes and their intersection is a $(d-1)$-dimensional constructible complex, then their union is constructible. In this definition, if in the recursion we restrict Δ_2 always to be a simplex, then the definition becomes equivalent to that of (pure) shellability. The notion of constructibility for simplicial complexes appears in [49]. It is known and easy to see that

Shellable $\Rightarrow$ constructible $\Rightarrow$ Cohen-Macaulay.

Since any shellable simplicial complex is partitionable (see [47, p. 79]), it is natural to ask whether any constructible complex is partitionable? This question is a special case of Stanley's conjecture that says that Cohen-Macaulay simplicial complexes are partitionable. We do not know the answer yet! In the following we present some examples where the complexes are not shellable or are not Cohen-Macaulay but the ideals related to these simplicial complexes are Stanley ideals.

Example 3.3.1. The following example of a simplicial complex is due to Masahiro Hachimori [19]. The simplicial complex Δ described by the next figure is 2-dimensional, non shellable but constructible. It is constructible, because if we divide the simplicial complex by the bold line, we obtain two shellable complexes, and their intersection is a shellable 1-dimensional simplicial complex.



Indeed we can write $\Delta = \Delta_1 \cup \Delta_2$ where the shelling order of the facets of Δ_1 is given by:

$$148, 149, 140, 150, 189, 348, 349, 378, 340, 390, 590, 569, 689, 678,$$

and that of Δ_2 is given by:

$$125, 126, 127, 167, 235, 236, 237, 356.$$

We use the following principle to construct a partition of Δ: suppose that Δ_1 and Δ_2 are d-dimensional partitionable simplicial complexes, and that $\Gamma = \Delta_1 \cap \Delta_2$ is $(d-1)$-dimensional pure simplicial complex. Let $\Delta_1 = \bigcup_{i=1}^{r}[K_i, L_i]$ be a nice partition of Δ_1, and $\Delta_2 = \bigcup_{i=1}^{s}[F_i, G_i]$ a nice partition of Δ_2. Suppose that for each i, the set $[F_i, G_i] \setminus \Gamma$ has a unique minimal element H_i. Then $\Delta_1 \cup \Delta_2 = \bigcup_{i=1}^{r}[K_i, L_i] \cup \bigcup_{i=1}^{s}[H_i, G_i]$ is a nice partition of $\Delta_1 \cup \Delta_2$. Notice that $[F_i, G_i] \setminus \Gamma$ has a unique minimal element if and only if for all $F \in [F_i, G_i] \cap \Gamma$ there exists a facet G of Γ with $F \subseteq G \subset G_i$.

Suppose that Δ_2 is shellable with shelling $G_1, \ldots, G_s$. Let F_i be the unique minimal subface of G_i which is not a subface of any G_j with $j < i$. Then $\Delta_2 = \bigcup_{i=1}^{s}[F_i, G_i]$ is the nice partition induced by this shelling. The above discussions then show that $\Delta_1 \cup \Delta_2$ is partitionable, if for all i and all $F \in \Gamma$ such that $F \subset G_i$ and $F \not\subset G_j$ for $j < i$, there exists a facet $G \in \Gamma$ with $F \subseteq G \subset G_i$.

In our particular case the shelling of Δ_1 induces the following partition of Δ_1:

$$[\emptyset, 148], [9, 149], [0, 140], [5, 150], [89, 189], [3, 348], [39, 349], [7, 378],$$

$$[30, 340], [90, 390], [59, 590], [6, 569], [68, 689], [67, 678],$$

and the shelling of Δ_2 induces the following partition of Δ_2:

$$[\emptyset, 125], [6, 126], [7, 127], [67, 167], [3, 235], [36, 236], [37, 237], [56, 356].$$

The facets of $\Gamma = \Delta_1 \cap \Delta_2$ are: $15, 56, 67, 73$.

The restriction of the intervals of this partition of Δ_2 to the complement of Γ do not all give intervals. For example we have $[6, 126] \setminus \Gamma = \{16, 26, 126\}$. This set has two minimal elements, and hence is not an interval. On the other hand, the following partition of Δ_2 (which is not induced from a shelling)

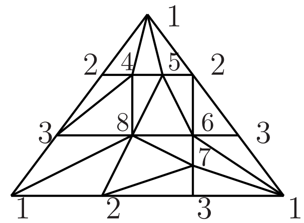
$$[\emptyset, 237], [1, 125], [5, 356], [6, 167], [17, 127], [25, 235], [26, 126], [36, 236]$$

restricted to the complement of Γ yields the following intervals

$$[2, 237], [12, 125], [35, 356], [16, 167], [17, 127], [25, 235], [26, 126], [36, 236],$$

which together with the intervals of the partition of Δ_1 give us a partition of Δ.

Example 3.3.2. (The Dunce hat) The Dunce hat is the topological space obtained from the solid triangle abc by identifying the oriented edges $\vec{ab}$, $\vec{bc}$ and $\vec{ac}$. The following is a triangulation of the Dunce hat using 8 vertices.



The facets arising from this triangulation are

$$124, 125, 145, 234, 348, 458, 568, 256, 236, 138, 128, 278, 678, 237, 137, 167, 136.$$

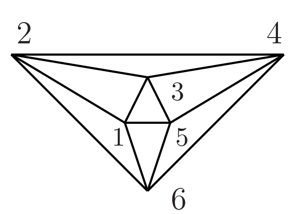
It is known that the simplicial complex corresponding to this triangulation is not shellable (not even constructible), but it is Cohen-Macaulay, see [19], and it has the following partition:

$$[\emptyset, 124], [3, 234], [5, 145], [6, 236], [7, 137], [8, 348], [13, 138], [16, 136], [18, 128],$$

$$[25, 125], [27, 237], [28, 278], [56, 256], [67, 167], [68, 568], [78, 678], [58, 458].$$

Therefore we have again $\operatorname{depth}(\Delta) = \dim(\Delta) = \operatorname{sdepth}(\Delta) = 3$.

Example 3.3.3. (The Cylinder) The ideal $I = (x_1x_4, x_2x_5, x_3x_6, x_1x_3x_5, x_2x_4x_6) \subset K[x_1, \ldots, x_6]$ is the Stanley-Reisner ideal of the triangulation of the cylinder shown in the next figure. The corresponding simplicial complex Δ is Buchsbaum but not Cohen-Macaulay.



The facets of Δ are $123, 126, 156, 234, 345, 456$, and it has the following partition:

$$[\emptyset, 123], [4, 234], [5, 345], [6, 456], [15, 156], [16, 126], [26, 26].$$

Therefore we have $\operatorname{depth}(\Delta) = \operatorname{sdepth}(\Delta) = 2 < 3 = \dim(\Delta)$. Although Δ is not partitionable, I_Δ is a Stanley ideal.

In Section 2.5 we show that the facet ideal I of any forest is clean and hence Stanley's conjecture holds for S/I. There is a more general class of simplicial complexes which is called quasi-forest. It is natural to ask whether the facet ideal of any quasi-forest is again clean?

According to [57], a connected simplicial complex Δ is called a *quasi-tree*, if there exists an order $F_1, \ldots, F_m$ of the facets, such that F_i is a leaf of $\langle F_1, \ldots, F_i \rangle$ for each $i = 1, \ldots, m$. Such an order is called a *leaf order*. A simplicial complex Δ with the property that every connected component is a quasi-tree is called a *quasi-forest*. It is clear that any forest is a quasi-forest.

Unfortunately the facet ideal of a quasi-forest need not to be clean. For example the facet ideal of the quasi-tree $\Gamma = \langle \{1, 2, 3, 4\}, \{1, 4, 5\}, \{1, 2, 8\}, \{2, 3, 7\}, \{3, 4, 6\} \rangle$, as in Figure 1, is not clean. Indeed

$$I(\Gamma)^\vee = (x_1x_3, x_2x_4, x_4x_7x_8, x_1x_6x_7, x_1x_4x_7, x_2x_3x_5, x_1x_2x_6, x_2x_5x_6, x_3x_4x_8, x_3x_5x_8)$$

has no linear quotients, even no componentwise linear quotients.

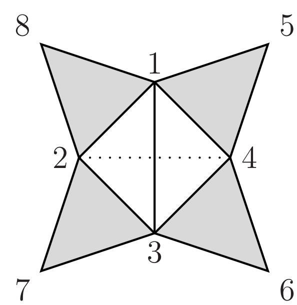


Figure 1:

One might expect that the facet ideal of any quasi-forest which is not a forest is not clean. The following example shows that this is not the case. The facet ideal of the quasi-tree $\Gamma' = \langle \{1, 2, 3\}, \{2, 4, 5\}, \{2, 3, 5\}, \{3, 5, 6\} \rangle$, as in Figure2, is clean. Since $I(\Gamma')^\vee = (x_3x_5, x_2x_5, x_1x_5, x_2x_6, x_2x_3, x_3x_4)$ has linear quotients in the given order.

It would be interesting to classify all quasi-forests such that their facet ideals are clean.

Even though $I(\Gamma)$ (Γ is the quasi-tree as given in Figure 1) is not clean we will show that Stanley's conjecture holds for $S/I(\Gamma)$. Indeed from the fact $I(\Gamma)^\vee =$

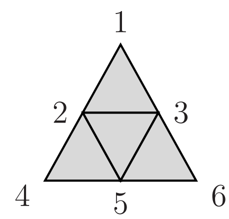


Figure 2:

$(I_\Delta)^\vee = I_{\Delta^\vee} = I(\Delta^c)$, it is easy to see that the facets of Δ are $\{1,3,5,6,7,8\}$, $\{2,4,5,6,7,8\}$, $\{1,2,3,5,6\}$, $\{2,3,4,5,8\}$, $\{2,3,5,6,8\}$, $\{1,4,6,7,8\}$, $\{3,4,5,7,8\}$, $\{1,3,4,7,8\}$, $\{1,2,5,6,7\}$, and $\{1,2,4,6,7\}$.

The partition $\mathcal{P} = [\emptyset, 135678] \cup [2, 12356] \cup [4, 245678] \cup [14, 14678] \cup [27, 12567] \cup [34, 34578] \cup [28, 23568] \cup [124, 12467] \cup [134, 13478] \cup [234, 23458] \cup [278, 25678]$ has the property that the cardinality of upper boundary of each interval in $\mathcal{P}$ is $\geq \min\{|F|\colon F \text{ is a facet of } \Delta\} \geq \operatorname{depth}(S/I_\Delta) = \operatorname{depth}(S/I(\Gamma))$.

4 Squarefree modules and Alexander duality

In this section we study how prime filtrations and squarefree Stanley decompositions of squarefree modules over the polynomial ring and over the exterior algebra behave with respect to Alexander duality.

4.1 Prime filtrations and Alexander duality

Let $S = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field K and M a finitely generated $\mathbb{Z}^n$-graded S-module. It is known that the associated prime ideals of M are monomial ideals, and any monomial prime ideal is of the form $P_F = (x_i \colon i \in F)$ for some $F \subset [n]$. A chain $0 = M_0 \subset M_1 \subset \dots \subset M_r = M$ of $\mathbb{Z}^n$-graded submodules of M such that $M_i/M_{i-1} \cong S/P_{F_i}(-G_i)$ is called a prime filtration of M. If M is a finitely generated $\mathbb{Z}^n$-graded S-module, then a prime filtration of M always exists, see [34, Theorem 6.4].

First we recall the definitions of squarefree S-modules and E-modules from Section 1.6. A finitely generated $\mathbb{N}^n$-graded S-module $M = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} M_{\mathbf{a}}$ is *squarefree* if the multiplication map $M_{\mathbf{a}} \to M_{\mathbf{a}+\varepsilon_i}$, $m \mapsto mx_i$, is bijective for all $\mathbf{a} \in \mathbb{N}^n$ and all $i \in \operatorname{supp}(\mathbf{a})$, and a finitely generated $\mathbb{N}^n$-graded E-module $N = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} N_{\mathbf{a}}$ is called squarefree if it has only squarefree components.

We denote by $SQ(S)$ the abelian category of the squarefree S-modules, where the morphisms are the $\mathbb{N}^n$-graded homogeneous homomorphisms, and denote by $SQ(E)$ the abelian category of squarefree E-modules, where the morphisms are the $\mathbb{N}^n$-graded homogeneous homomorphisms. Tim Römer [39, Corollary 1.6] proved that there are two exact additive covariant functors

$$\mathbf{F} \colon SQ(S) \mapsto SQ(E), \quad M \mapsto \mathbf{F}(M) \quad \text{and} \quad \mathbf{G} \colon SQ(E) \mapsto SQ(S), \quad N \mapsto \mathbf{G}(N)$$

of abelian categories such that $(\mathbf{F} \circ \mathbf{G})(N) = N$ and $(\mathbf{G} \circ \mathbf{F})(M) = M$. Hence the categories $SQ(S)$ and $SQ(E)$ are equivalent.

We shall need the following:

Lemma 4.1.1. *Let $M \subset M'$ be two squarefree S-modules and $N \subset N'$ be two squarefree E-modules.*

(a) *If $M'/M \cong S/P_F(-G)$, then $G \cap F = \emptyset$;*

(b) *We have $M'/M \cong S/P_F(-G)$ if and only if $\mathbf{F}(M')/\mathbf{F}(M) \cong E/P_{F\cup G}(-G)$, where $P_{F\cup G} = (e_j \colon j \in F \cup G)$;*

(c) *We have $N'/N \cong E/P_{F\cup G}(-G)$ if and only if $\mathbf{G}(N')/\mathbf{G}(N) \cong S/P_F(-G)$.*

Proof. (a) Suppose $G \cap F \neq \emptyset$. Let $i \in G \cap F$ and let f the homogeneous generator of M'/M. Since M'/M is squarefree, and since $\deg f = G$ it it follows that $x_i f \neq 0$, a contradiction.

(b) Since $\mathbf{F}$ is an exact functor it suffices to show that $\mathbf{F}(S/P_F(-G)) = E/P_{F\cup G}(-G)$. But this follows immediately from the Aramova-Avramov-Herzog complex [1, Theorem 1.3] by which Römer defined the functor $\mathbf{F}$ in [39].

(c) follows form (b) by using the fact that the functors $\mathbf{F}$ and $\mathbf{G}$ are inverse to each other. $\square$

If $M \subset M'$ are two squarefree S-modules such that $M'/M \cong S/P_F(-G)$, then we have the following short exact sequence

$$0 \to M \to M' \to M'/M \cong S/P_F(-G) \to 0.$$

Applying Lemma 4.1.1 to the above short exact sequence we get the following short exact sequence

$$0 \to \mathbf{F}(M) \to \mathbf{F}(M') \to E/P_{F\cup G}(-G) \to 0.$$

Since $\mathrm{Hom}_E(-, E)$ is an contravariant exact functor, from the above short exact sequence we obtain the short exact sequence

$$0 \to \mathrm{Hom}_E(E/P_{F\cup G}(-G), E) \to \mathbf{F}(M')^\vee \to \mathbf{F}(M)^\vee \to 0.$$

On the other hand $\mathrm{Hom}_E(E/P_{F\cup G}(-G), E) = \mathrm{Hom}_E(E/P_{F\cup G}, E)(G)$. Since

$$\mathrm{Hom}_E(E/P_{F\cup G}, E) = 0 :_E P_{F\cup G} = (e_{F\cup G}) \cong E/P_{F\cup G}(-F-G),$$

one has $\mathrm{Hom}_E(E/P_{F\cup G}(-G), E) \cong E/P_{F\cup G}(-F)$.

We conclude that the natural map

$$\alpha\colon \mathbf{F}(M')^\vee \to \mathbf{F}(M)^\vee$$

is an epimorphism with $\mathrm{Ker}(\alpha) \cong E/P_{F\cup G}(-F)$.

Proposition 4.1.2. *Let N be a squarefree E-module and $N^\vee$ its E-dual. Then there exists a chain $0 \subset N_1 \subset \ldots \subset N_t = N$ of squarefree submodules of N with $N_i/N_{i-1} \cong E/P_{F_i\cup G_i}(-G_i)$ if and only if there exists a chain $0 \subset H_1 \subset \ldots \subset H_t = N^\vee$ of squarefree submodule of $N^\vee$ with $H_i/H_{i-1} \cong E/P_{F_i\cup G_i}(-F_i)$.*

Proof. It is enough to prove one direction of the assertion, because $(N^\vee)^\vee = N$. Let $0 = N_0 \subset N_1 \subset \ldots \subset N_t = N$ be a chain of squarefree E-modules with $N_i/N_{i-1} \cong E/P_{F_i\cup G_i}(-G_i)$. From the observation above we see that for each i there is an epimorphism $\alpha_i\colon N^\vee_{t-i+1} \to N^\vee_{t-i}$ with $\mathrm{Ker}\,\alpha_i \cong E/P_{F_i\cup G_i}(-F_i)$.

Let $\beta_i\colon N^\vee \to N^\vee_{t-i}$ be the epimorphism which is defined by $\beta_i = \alpha_i \circ \alpha_{i-1} \circ \cdots \circ \alpha_1$. Then

$$0 \subset \mathrm{Ker}\,\beta_1 \subset \cdots \subset \mathrm{Ker}\,\beta_t = N^\vee$$

is a filtration of $N^\vee$ by squarefree E-modules. We only need to show that $\mathrm{Ker}\,\beta_i/\mathrm{Ker}\,\beta_{i-1} \cong \mathrm{Ker}\,\alpha_i$. This follows from the Snake Lemma applied to the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ker}\beta_{i-1} & \xrightarrow{\iota_1} & N^\vee & \xrightarrow{\beta_{i-1}} & N^\vee_{t-i+1} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \iota_2} & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle \alpha_i} & & \\ 0 & \longrightarrow & \operatorname{Ker}\beta_i & \xrightarrow{\iota_3} & N^\vee & \xrightarrow{\beta_i} & N^\vee_i & \longrightarrow & 0 \end{array}$$

with exact rows, where the ι_j are inclusion maps. □

Now we can prove the corresponding result for squarefree S-modules.

Theorem 4.1.3. *Let M be a squarefree S-module and $M^\vee$ its Alexander dual. Then there exists a chain $0 \subset M_1 \subset \cdots \subset M_r = M$ of squarefree submodules of M with $M_i/M_{i-1} \cong S/P_{F_i}(-G_i)$ if and only if there exists a chain $0 \subset L_1 \subset \cdots \subset L_r = M^\vee$ of squarefree submodules of $M^\vee$ with $L_i/L_{i-1} \cong S/P_{G_i}(-F_i)$.*

Proof. Again it is enough to prove one direction of the assertion, because $(M^\vee)^\vee = M$. From the given chain of submodules of M we get a chain

$$0 \subset \mathbf{F}(M_1) \subset \cdots \subset \mathbf{F}(M_r) = \mathbf{F}(M)$$

of squarefree E-modules with $\mathbf{F}(M_i)/\mathbf{F}(M_{i-1}) \cong E/P_{F_i\cup G_i}(-G_i)$, see Lemma 4.1.1(b). Therefore by Proposition 4.1.2 there exists a chain $0 \subset N_1 \subset \cdots \subset N_{r-1} \subset N_r = (\mathbf{F}(M))^\vee$ of squarefree E-modules with $N_i/N_{i-1} \cong E/P_{F_i\cup G_i}(-F_i)$. This chain of squarefree E-modules induces the chain

$$0 \subset \mathbf{G}(N_1) \subset \cdots \subset \mathbf{G}(N_{r-1}) \subset \mathbf{G}(N_r) = \mathbf{G}(\mathbf{F}(M))^\vee) = M^\vee$$

of squarefree S-modules with $\mathbf{G}(N_i)/\mathbf{G}(N_{i-1}) \cong S/P_{G_i}(-F_i)$, see Lemma 4.1.1(c). □

We now explain what Theorem 4.1.3 means in the special case that $M = J/I$ where $I \subset J \subset S$ are squarefree monomial ideals. To this end we introduce the following notation: let $I \subset S$ be a squarefree monomial ideal and Δ be the simplicial complex such that $I = I_\Delta$. We set $\tilde{I} = I_{\Delta^\vee}$. Then $\tilde{\tilde{I}} = I$ since $(\Delta^\vee)^\vee = \Delta$, and if $I \subset J$ are two squarefree monomial ideals, then $\tilde{J} \subset \tilde{I}$ and $(J/I)^\vee = \tilde{I}/\tilde{J}$.

Corollary 4.1.4. *Let $I \subset J$ be a squarefree monomial ideals. The following conditions are equivalent:*

(a) *$I = I_0 \subset I_1 \subset \ldots \subset I_{r-1} \subset I_r = J$ is an $\mathbb{N}^n$-graded prime filtration of J/I with $I_i/I_{i-1} \cong S/P_{F_i}(-G_i)$.*

(b) *$\tilde{J} = \tilde{I}_r \subset \tilde{I}_{r-1} \subset \ldots \subset \tilde{I}_1 \subset \tilde{I}_0 = \tilde{I}$ is an $\mathbb{N}^n$-graded prime filtration of $\tilde{I}/\tilde{J} = (J/I)^\vee$ with $\tilde{I}_{i-1}/\tilde{I}_i \cong S/P_{G_i}(-F_i)$.*

Proof. It is enough to prove the implication (a) $\Rightarrow$ (b), because $\tilde{\tilde{L}} = L$ for any squarefree monomial ideal L. For the proof we may assume that $r = 1$, in other words $J/I \cong S/P_F(-G)$. In this situation $\tilde{I}/\tilde{J} = (J/I)^\vee \cong S/P_G(-F)$, by Theorem 4.1.3. □

4.2 Stanley decompositions of squarefree modules and Alexander duality

In this section we study squarefree Stanley decompositions of squarefree S-modules and E-modules. We also show how squarefree Stanley decompositions behave with respect to Alexander duality. For this first we generalized some notation and results from Section 3.2 about squarefree modules of type S/I, where I is a qsquarefree monomial ideal, to squarefree S-modules in general.

A Stanley space $mK[Z]$ is called *squarefree* if m is a squarefree homogeneous element and $\mathrm{supp}(m) \subset \mathrm{supp}(Z) = \{i \colon x_i \in Z\}$. The Stanley decomposition $\mathcal{D}$ of M is called a *squarefree Stanley decomposition* if all Stanley spaces in $\mathcal{D}$ are squarefree Stanley spaces. For a squarefree module M we denote by

$$\mathrm{sqdepth}(M) = \max\{\mathrm{sdepth}(\mathcal{D}) \colon \mathcal{D} \text{ is a squarefree Stanley decomposition of } M\}$$

the *squarefree Stanley depth* of M. It is clear that $\mathrm{sqdepth}(M) \leq \mathrm{sdepth}(M)$.

As a generalization of Lemma 3.2.1 we have the following. The argument of the proof is similar, nevertheless her we also give the proof.

Proposition 4.2.1. *Let M be a finitely generated $\mathbb{N}^n$-graded S-module. Then M has a squarefree Stanley decomposition if and only if M is a squarefree S- module.*

Proof. Let M be a suarefree S-module. Then by [56, Proposition 2.5] M has a filtration of $\mathbb{N}^n$-graded submodules $0 \subset M_1 \subset \ldots \subset M_r = M$ of M such that each quotient $M_i/M_{i-1} \cong S/P_{F_i^c}(-F_i)$ for some $F_i \subset [n]$ where $F_i^c = [n] \setminus F_i$. Hence by 3.1.1 $M = \bigoplus_{i=1}^r m_i K[Z_{F_i}]$ is asquarefree Stanley decomposition of M, where $m_i \in M_{F_i}$ is a squarefree element.

For the converse assume that $M = \bigoplus_{i=1}^r m_i K[Z_i]$ is a squarefree Stanley decomposition of M. We will show that the multiplication map $M_{\mathbf{a}} \to M_{\mathbf{a}+\varepsilon_i}$, $m \mapsto mx_i$, is bijective for all $\mathbf{a} \in \mathbb{N}^n$ and all $i \in \mathrm{supp}(\mathbf{a})$. Let $\mathbf{a} \in \mathbb{N}^n$. First we show if $M_{\mathbf{a}} = 0$, than $M_{\mathbf{a}+\varepsilon_i} = 0$ for any $i \in \mathrm{supp}(\mathbf{a})$. Suppose that $M_{\mathbf{a}+\varepsilon_i} \neq 0$ and $0 \neq m \in M_{\mathbf{a}+\varepsilon_i}$. Then there exists some $1 \leq t \leq r$ such that $m \in m_t K[Z_t]$. This implies that $m = m_t x_F$ for some $F \subset \mathrm{supp}(\mathbf{a})$ with $i \in F$. Since $m_t K[Z_t]$ is a Stanley space, one has $m_i(x_F/x_i)$ is a non zero element in $m_t K[Z_t]$. This is a contradiction, since $m_i(x_F/x_i) \in M_{\mathbf{a}}$.

Now assume that $M_{\mathbf{a}} \neq 0$. Let $0 \neq m \in M_{\mathbf{a}}$. Then there exists some $1 \leq t \leq r$ such that $m \in m_t K[Z_t]$. Since $m_t K[Z_t]$ is a squarefree Stanley space we have $\mathrm{supp}(m) = \mathrm{supp}(\mathbf{a}) \subset \mathrm{supp}(Z_t)$ and hence $mx_i \neq 0$ for any $i \in \mathrm{supp}(\mathbf{a})$. This shows that the map is injective. For surjection let $0 \neq m \in M_{\mathbf{a}+\varepsilon_i}$. Again there exists some $1 \leq t \leq r$ such that $m \in m_t K[Z_t]$. This implies that $m = m_t x_F$ for some $F \subset \mathrm{supp}(\mathbf{a})$ with $i \in F$. Hence $m = m_i(x_F/x_i)x_i$ where $m_i(x_F/x_i) \in M_{\mathbf{a}}$. $\square$

Again as a generalization of Theorem 3.2.3 we have the following.

Theorem 4.2.2. *Let M be an $\mathbb{N}^n$-graded squarefree S-module. Then*

$$\mathrm{sqdepth}(M) = \mathrm{sdepth}(M).$$

Proof. Let $\mathcal{D}$ be a Stanley decomposition of M and m a homogeneous element in M. Since M is squarefree, there is a squarefree element $m' \in M$ such that $\mathrm{supp}(m') = \mathrm{supp}(m)$ and $m = m'x_F$ for some $F \subset \mathrm{supp}(m)$. Hence there exists a summand $nK[Z]$ in $\mathcal{D}$ with $m' \in nK[Z]$. This implies that n is squarefre and

$$\mathrm{supp}(n) \subset \mathrm{supp}(m') = \mathrm{supp}(m).$$

Set $Z' = Z \cup \{x_i \colon i \in \mathrm{supp}(n)\}$. Then $\mathrm{supp}(n) \subset \{j \colon x_j \in Z'\}$. Hence $nK[Z']$ is a squarefree Stanley space and $m \in nK[Z']$. Let $\mathcal{D}'$ be the sum of those Stanley spaces $nK[Z']$, where $nK[Z]$ is in $\mathcal{D}$ and n is squarefree. This sum is direct. Indeed, let $u \in n_iK[Z_i'] \cap n_jK[Z_j']$. We may assume that u is squarefrre, since $n_iK[Z_i']$ and $n_jK[Z_j']$ are squarefree. Therefore $u \in n_iK[Z_i] \cap n_jK[Z_j]$, which is a contradiction. On the other hand this sum cover M, as we show above it covers all homogeneous elements of M. Therefore $\mathcal{D}'$ is a Stanley decomposition of M with $\mathrm{sdepth}(\mathcal{D}') \geq \mathrm{sdepth}(\mathcal{D})$. This shows that $\mathrm{sqdepth}(M) \geq \mathrm{sdepth}(M)$. The other inequality is obvious. $\square$

Let $E = K\langle e_1, \ldots, e_n\rangle$ be the exterior algebra over an n-dimensional K-vector space V and N a finitely generated $\mathbb{N}^n$-graded E-module. Let $n \in N$ be a homogeneous element and $A \subset \{e_1, \ldots, e_n\}$. We set $\mathrm{supp}(n) = \mathrm{supp}(\deg(n))$ and $\mathrm{supp}(A) = \{j \colon e_j \in A\}$. We denote by $nK\langle A\rangle$ the the K-subspace of N generated by all homogeneous elements of the form ne_F, where $e_F \in K\langle A\rangle$. If the elements ne_F with $F \in \mathrm{supp}(A)$ form a K-basis of $nK\langle A\rangle$, then we call $nK\langle A\rangle$ a *Stanley space of dimension* $|A|$.

In case N is a squarefree and $nK\langle A\rangle \subset N$ is a Stanley space we have that $\mathrm{supp}(n)$ is squarefree and $\mathrm{supp}(n) \cap \mathrm{supp}(A) = \emptyset$. A direct sum $N = \bigoplus_{i=1}^t n_iK\langle A_i\rangle$ with Stanley spaces $n_iK\langle A_i\rangle$ is called a *Stanley decomposition* of N.

Proposition 4.2.3. *Let N be a squarefree E-module, and $N^\vee$ the E-dual of N. Then there exists a Stanley decomposition $N = \bigoplus_{i=1}^t n_iK\langle A_i\rangle$ of N if and only if there exists a Stanley decomposition $N^\vee = \bigoplus_{i=1}^t b_iK\langle A_i\rangle$ of $N^\vee$ with*

$$\mathrm{supp}(b_i) = [n] \setminus (\mathrm{supp}(A_i) \cup \mathrm{supp}(n_i)).$$

Proof. By Theorem 1.6.12 we have $N^\vee \cong N^* = \mathrm{Hom}_K(N, K(-\mathbf{1}))$. Hence we will show the assertion for N^*. Since $N = \bigoplus_{i=1}^t n_iK\langle A_i\rangle$, as an $\mathbb{N}^n$-graded K-vector space one has $N^* = \bigoplus_{i=1}^t (n_iK\langle A_i\rangle)^*$. Set $\mathrm{supp}(n_i) = F_i$ and $\mathrm{supp}(A_i) = G_i$. Then $F_i \cap G_i = \emptyset$ and the elements n_ie_H with $H \subseteq G_i$ form a K-basis of $n_iK\langle A_i\rangle$. Consequently, the dual elements $(n_ie_H)^*$ form a K-basis of $(n_iK\langle A_i\rangle)^*$.

Let $b_i = (n_ie_{G_i})^*$ and $H, L \subseteq G_i$. Then

$$(b_ie_H)(n_ie_L) = \pm b_i(n_ie_Le_H) = \begin{cases} 0, & \text{if} \quad L \neq G_i \setminus H, \\ \pm 1, & \text{if} \quad L = G_i \setminus H, \end{cases}$$

and for any $j \neq i$ and all $T \subset G_j$ one has $(b_ie_H)(n_je_T) = \pm b_i(n_je_Te_H) = 0$. This shows that $b_ie_H = \pm(n_ie_{G_i \setminus H})^*$ for any $H \subset G_i$. Therefore $(n_iK\langle A_i\rangle)^* = b_iK\langle A_i\rangle$ and $N^* = \bigoplus_{i=1}^t b_iK\langle A_i\rangle$. $\square$

Let M be a squarefree S-module and let N be its corresponding squarefree E-module. In Section 1.6 we showed that there is an isomorphism $\theta\colon M_{\text{sq}} \to N$ of graded K-vector spaces. We will use this isomorphism to describe in the next lemma the relationship between squarefree Stanley decompositions of M and Stanley decompositions of N.

Lemma 4.2.4. (a) *Let $M = \bigoplus_{i=1}^{t} m_i K[Z_i]$ be a squarefree Stanley decomposition of M and*

$$A_i = \{e_j \colon j \in \operatorname{supp}(Z_i) \setminus \operatorname{supp}(m_i)\}.$$

Then $N = \bigoplus_{i=1}^{t} n_i K\langle A_i\rangle$ is a Stanley decomposition of N, where $n_i = \theta(m_i) \in N$ for $i = 1, \ldots, t$.
(b) *Conversely, if $N = \bigoplus_{i=1}^{t} n_i K\langle A_i\rangle$ is a Stanley decomposition of N and*

$$Z_i = \{x_j \colon j \in \operatorname{supp}(A_i) \cup \operatorname{supp}(n_i)\}.$$

Then $M = \bigoplus_{i=1}^{t} m_i K[Z_i]$ is a squarefree Stanley decomposition of M, where $m_i = \theta^{-1}(n_i) \in M$ for $i = 1, \ldots, t$.

Proof. (a): Since $M = \bigoplus_{i=1}^{t} m_i K[Z_i]$, one has

$$\bigcup_{i=1}^{t} \{m_i x_F \colon F \subset \operatorname{supp}(A_i)\}$$

forms a K-basis of M_{sq}, and hence

$$\bigcup_{i=1}^{t} \{\theta(m_i x_F) \colon F \subset \operatorname{supp}(A_i)\}$$

forms a K-basis of N. By Lemma 1.6.10 we have $\theta(m_i x_F) = (-1)^{\sigma(G_i,F)} n_i e_F$, where $G_i = \operatorname{supp}(m_i)$. Therefore

$$\bigcup_{i=1}^{t} \{n_i e_F \colon F \subset \operatorname{supp}(A_i)\}$$

forms a K-basis of N.

(b): Let $x^{\mathbf{a}} \in K[Z_i]$. We can write $x^{\mathbf{a}} = x^{\mathbf{a}'} x^{\mathbf{b}}$ where $\mathbf{b} \in \mathbb{N}^n$ is a squarefree vector with $F = \operatorname{supp}(\mathbf{b}) \subset \operatorname{supp}(A_i)$. Then

$$m_i x^{\mathbf{a}} = (m_i x^{\mathbf{b}}) x^{\mathbf{a}'} = (-1)^{\sigma(G_i,F)} \theta^{-1}(n_i e_F) x^{\mathbf{a}'}.$$

Since $\theta^{-1}(n_i e_F) \neq 0$ and since M is squarefree and $\operatorname{supp}(\mathbf{a}') \subset \operatorname{supp}(\theta^{-1}(n_i e_F))$, one has $m_i x^{\mathbf{a}} \neq 0$. Therefore

$$\bigcup_{i=1}^{t} \{m_i x^{\mathbf{a}} \colon x^{\mathbf{a}} \in K[Z_i]\}$$

forms a K-basis of M. □

Corollary 4.2.5. *Let N be an $\mathbb{N}^n$-graded E-module. Then N is squarefree if and only if*

$$N = \bigoplus_{i=1}^{t} n_i K\langle A_i\rangle$$

is a Stanley decomposition of N with $\operatorname{supp}(n_i) \cap \operatorname{supp}(A_i) = \emptyset$.

Now we will present the main result of this section.

Theorem 4.2.6. *Let M be a squarefree S-module, and $M^\vee$ its Alexander dual. Then there exists a squarefree Stanley decomposition $M = \bigoplus_{i=1}^{t} m_i K[Z_i]$ of M if and only if there exists a squarefree Stanley decomposition $M^\vee = \bigoplus_{i=1}^{t} v_i K[W_i]$ of $M^\vee$ with* $\operatorname{supp}(v_i) = [n] \setminus \operatorname{supp}(Z_i)$ *and* $W_i = \{x_j \colon j \in [n] \setminus \operatorname{supp}(m_i)\}$.

Proof. Let $M = \bigoplus_{i=1}^{t} m_i K[Z_i]$ be a squarefree Stanley decomposition of M. If we set $F_i = \operatorname{supp}(m_i)$ and $G_i = \operatorname{supp}(Z_i) \setminus F_i$, then $F_i \cap G_i = \emptyset$. Let N be the squarefree E-module corresponding to M. Then by Lemma 4.2.4(a), N has a Stanley decomposition

$$N = \bigoplus_{i=1}^{t} n_i K\langle A_i\rangle$$

where $n_i = \theta(m_i)$ and $G_i = \operatorname{supp}(A_i)$. Hence by Proposition 4.2.3, $N^\vee$ has a decomposition $N^\vee = \bigoplus_{i=1}^{t} b_i K\langle A_i\rangle$ with $\operatorname{supp}(b_i) = [n] \setminus (G_i \cup F_i)$. Therefore by Lemma 4.2.4(b), $M^\vee$ the corresponding squarefree S-module to $N^\vee$ has a decomposition as required. $\square$

Associated to any finitely generated $\mathbb{N}^n$-graded S-module M is a *minimal $\mathbb{Z}^n$-graded free resolution*

$$0 \to \bigoplus_j S(-\mathbf{a}_j)^{\beta_{r,j}(M)} \to \cdots \to \bigoplus_j S(-\mathbf{a}_j)^{\beta_{1,j}(M)} \to \bigoplus_j S(-\mathbf{a}_j)^{\beta_{0,j}(M)} \to 0$$

where $S(-\mathbf{a}_j)$ denote the $\mathbb{Z}^n$-graded S-module obtained by shifting the degrees of S by $\mathbf{a}_j$. The number $\beta_{i,j}(M)$ is the *ij-th graded Betti number* of M. The regularity of M is

$$\operatorname{reg}(M) = \max\{|\mathbf{a}_j| - i \colon \text{for all } i, j\}.$$

Let M be a squarefree $\mathbb{N}^n$-graded S-module. If Stanley's conjecture holds for M, then by Theorem 4.2.2 we may assume that there exists a squarefree Stanley decomposition $M = \bigoplus_{i=1}^{t} m_i K[Z_i]$ of M such that $|Z_i| \geq \operatorname{depth}(M)$. Also by Theorem 4.2.6 there exists a squarefree Stanley decomposition $M^\vee = \bigoplus_{i=1}^{t} v_i K[W_i]$ of the Alexander dual of M such that $|\deg(v_i)| = n - |Z_i| \leq n - \operatorname{depth}(M)$. On the other hand $\operatorname{proj\,dim}(M) = \operatorname{reg}(M^\vee)$, see [40, Corollary 3.7]. Since $\operatorname{depth}(M) + \operatorname{proj\,dim}(M) = n$, see [7, Theorem 1.3.3], we have $|\deg(v_i)| \leq \operatorname{reg}(M^\vee)$ for all i. Therefore we will get the following:

Corollary 4.2.7. *Let M be a squarefree $\mathbb{N}^n$-graded S-module and $M^\vee$ its Alexander dual. Then Stanley's conjecture holds for M if and only if $M^\vee$ has a squarefree Stanley decomposition $M^\vee = \bigoplus_{i=1}^t v_i K[W_i]$ with $|\deg(v_i)| \leq \operatorname{reg}(M^\vee)$ for all i.*

In the case that $I \subset S$ is a monomial ideal and $M = S/I$ or $M = I$, then we may consider the standard grading for S and M by setting $\deg(x_i) = 1$ for $i = 1, \ldots, n$. In this case a minimal graded free resolution of I is

$$0 \to \bigoplus_j S(-j)^{\beta_{r,j}(M)} \to \cdots \to \bigoplus_j S(-j)^{\beta_{1,j}(M)} \to \bigoplus_j S(-j)^{\beta_{0,j}(M)} \to M \to 0.$$

The regularity of M in this case is

$$\operatorname{reg}(M) = \max\{j - i \colon \text{for all} \quad i, j\}.$$

Suppose that all monomial minimal generators of I are of degree d. Then I has a *linear resolution* if for all $i \geq 0$, $\beta_{i,j} = 0$ for all $j \neq i + d$. In this situation $\operatorname{reg}(I) = d$.

In [47] Stanley conjectured that any Cohen-Macaulay simplicial complex is partitionable, see also [48]. In Section 3.2 we shoe that this conjecture is a special case of Stanley's conjecture on Stanley decompositions. Indeed we show that if

$$\mathbf{P} : \ \Delta = \bigcup_{i=1}^t [F_i, G_i]$$

is a partition of Δ, i.e $\mathcal{F}(\Delta) = \{G_1, \ldots, G_t\}$, then $\mathcal{D}(\mathbf{P}) : \ S/I_\Delta = \bigoplus x_{F_i} K[Z_{G_i}]$ is a squarefree Stanley decomposition of S/I_Δ, where $x_{F_i} = \prod_{j \in F_i} x_j$ and $Z_{G_i} = \{x_j \colon j \in G_i\}$. Hence we get the following corollary.

Corollary 4.2.8. *A Cohen-Macaulay simplicial complex Δ is partitionable if and only if $I_{\Delta^\vee}$ has a squarefree Stanley decomposition $I_{\Delta^\vee} = \bigoplus_{i=1}^t u_i K[Z_i]$ such that $\{u_i, \ldots, u_t\} = G(I_{\Delta^\vee})$.*

Proof. By Eagon-Reiner [15] Δ is Cohen-Macaulay if and only if $I_{\Delta^\vee}$ has a linear resolution. Also by a result of Terai [52] $\operatorname{proj\,dim}(S/I_\Delta) = \operatorname{reg}(I_{\Delta^\vee})$ for any simplicial complex Δ.

On the other hand by Corollary 4.2.7 the Cohen-Macaulay simplicial complex Δ is partitionable if and only if $I_{\Delta^\vee}$ has a squarefree Stanley decomposition $I_{\Delta^\vee} = \bigoplus_{i=1}^t u_i K[Z_i]$ such that $\deg u_i \leq \operatorname{reg}(I_{\Delta^\vee}) = d$, where d is the degree of any minimal monomial generator of $I_{\Delta^\vee}$. Since $u_i \in I_{\Delta^\vee}$, one has $\deg(u_i) \geq d$ for all i. This shows that $u_i \in G(I_{\Delta^\vee})$ and hence $\{u_i, \ldots, u_t\} \subset G(I_{\Delta^\vee})$. The other inclusion is obvious. $\square$

Corollary 4.2.8 shows that Stanley's conjecture which says that any Cohen-Macaulay simplicial complex is patitionable is equivalent to say that any squarefree monomial ideal $I \subset S$ which has a linear resolution has a Stanley decomposition $I = \bigoplus_{i=1}^t u_i K[Z_i]$ such that $\{u_1, \ldots, u_t\} = G(I)$.

This results lead us to make the following conjecture which in the case of squarefree $\mathbb{N}^n$-graded S-module is equivalent to Stanley's conjecture on Stanley decompositions.

Conjecture 4.2.9. *Let $S = K[x_1, \dots, x_n]$, and let M be a finitely generated $\mathbb{Z}^n$-graded S-module. Then there exists a Stanley decomposition*

$$M = \bigoplus_{i=1}^{t} m_i K[Z_i],$$

of M with $|m_i| \leq \operatorname{reg} M$ for all i.

Let $\mathcal{D}$ be a Stanley decomposition of M. We call the maximal $|m_i|$ in $\mathcal{D}$ the *h-regularity* of $\mathcal{D}$, and denote it by $\operatorname{hreg}(\mathcal{D})$. Maclagan and Smith [36, Remark 4.2] proved that $\operatorname{hreg}(\mathcal{D}) \geq \operatorname{reg}(M)$ in the case that $M = S/I$, where I is a monomial ideal, and $\mathcal{D}$ is a Stanley filtration. We set

$$\operatorname{hreg}(M) = \min\{\operatorname{hreg}(\mathcal{D})\colon \mathcal{D} \text{ is a Stanley decomposition of } M\},$$

and call this number the *h-regularity* of M. With the notation introduced our conjecture says that $\operatorname{hreg}(M) \leq \operatorname{reg}(M)$.

Let M be a finitely generated $\mathbb{N}^n$-graded S-module which is generated by homogeneous elements $n_1, \dots, n_s$. It is clear that $|n_i| \leq \operatorname{reg}(M)$ for $i = 1, \dots, s$. We want to show that $|n_i| \leq \operatorname{hreg}(M)$ for $i = 1, \dots, s$. Let $\mathcal{D} = \bigoplus_{i=1}^{t} m_i K[Z_i]$ be a Stanley decomposition of M such that $\operatorname{hreg}(\mathcal{D}) = \operatorname{hreg}(M)$, and $|n_r| = \max\{|n_i|\colon i = 1, \dots, s\}$. Since $n_r \in M$ is a homogeneous element, there exists a $j \in [t]$ such that $n_r \in m_j K[Z_j]$. On the other hand $m_j \in M$ and n_r is a generator. Therefore we have $m_j = n_r$ and $|n_r| = |m_j| \leq \operatorname{hreg}(\mathcal{D})$.

It is known that Stanley's conjecture is true in some cases. If we combine some of this cases with Corollary 4.2.7 we get

Corollary 4.2.10. *Let $I = I_\Delta \subset S = K[x_1, \dots, x_n]$ be a squarefree monomial ideal. Then Conjecture 4.2.9 holds for*

(i) *I if $\deg(u) \geq n-1$ for all $u \in G(I)$;*

(ii) *I and S/I if $n \leq 3$;*

(iii) *I if I is a matroidal ideal of a transversal matroid;*

(iv) *S/I if $\operatorname{reg}(S/I) \geq n-2$;*

(v) *I if $\operatorname{reg}(I) = 2$,*

(vi) *I if I has a linear resolution and $n \leq 4$;*

Proof. (i): Since $\deg(u) \geq n-1$ for any $u \in G(I_\Delta)$, one has $\operatorname{height}(I_{\Delta^\vee}) \geq n-1$. Hence by Theorem 3.1.3 (a) Stanley's conjecture is true for $S/I_{\Delta^\vee}$ and assertion follows by Corollary 4.2.7.

(ii): It follows from Corollary 4.2.7, since Stanley's conjecture is true for $I_{\Delta^\vee}$ by [5], and for $S/I_{\Delta^\vee}$ by Theorem 3.1.3 (b) if $n \leq 3$.

(iii): It is obvious since in this case $I_{\Delta^\vee}$ is complete intersection monomial ideal and by Theorem 3.1.3 (c) Stanley's conjecture holds for $S/I_{\Delta^\vee}$.

(iv): Suppose that $\operatorname{reg}(S/I_\Delta) \geq n-2$. Then by a result of Terai [52] we have $\operatorname{proj\,dim}(I_{\Delta^\vee}) \geq n-2$. Hence $\operatorname{depth}(I_{\Delta^\vee}) \leq 2$ and by [5] Stanley's conjecture is true for $I_{\Delta^\vee}$. Therefore by Corollary 4.2.7 we are done.

(v): By Eagon-Reiner [15] $S/I_{\Delta^\vee}$ is Cohen–Macaulay. Hence by Theorem 3.1.3 (e) Stanley's conjecture holds for $S/I_{\Delta^\vee}$. Therefore by Corollary 4.2.7 we are done. □

Let $I = (u_1, \ldots, u_m)$ be a monomial ideal in S. According to [32], the monomial ideal I has linear quotients if one can order the set of minimal generators of I, $G(I) = \{u_1, \ldots, u_m\}$, such that the ideal $(u_1, \ldots, u_{i-1}) : u_i$ is generated by a subset of the variables for $i = 2, \ldots, m$.

Assume that $I = (u_1, \ldots, u_m)$ is a monomial ideal which has linear quotients with respect to the given order. Set $I_i = (u_1, \ldots, u_{i-1}) : u_i$, $Z_i = X \setminus G(I_i)$ for $i = 2, \ldots, m$ and $Z_1 = X$. We denote $r_i = |G(I_i)|$ for $i = 2, \ldots, m$ and $r(I) = \max\{r_i \colon i = 2, \ldots, s\}$. By [21, page 539] $\operatorname{depth}(I) = n - r(I)$.

Corollary 4.2.11. *Let $I \subset S$ be a monomial ideal with linear quotients. Then Stanley's conjecture on Stanley decompositions holds for I.*

Proof. Suppose $I = (u_1, \ldots, u_m)$ has linear quotients with respect to the given order. Then $\mathcal{G}:\ (0) \subset J_1 = (u_1) \subset \ldots \subset J_{m-1} = (u_1, \ldots, u_{m-1}) \subset J_m = I$ is a prime filtration of I. Hence by Proposition 3.1.1 $\mathcal{D} = \bigoplus_{i=1}^{s} u_i K[Z_i]$ is a Stanley decomposition of I with $\operatorname{sdepth}(\mathcal{D}) = n - r(I) = \operatorname{depth}(I)$. □

In the decomposition above of I, all u_i are the minimal monomial generators of I. Therefore we have

Corollary 4.2.12. *If $I \subset S$ is a monomial ideal which has linear quotient, then Conjecture 4.2.9 holds for I.*

In [27] it was shown that if I is monomial ideal with 2-linear resolution, then I has linear quotients. For example the Stanley–Reisner ideal of any forest and quasi forest has 2-linear resolution. Therefore Stanley's conjecture on Stanley decompositions and Conjecture 4.2.9 holds for such monomial ideals.

5 Monomial ideals with linear quotients

Let K be a field, $S = K[x_1, \ldots, x_n]$ the polynomial ring in n variables, and $I \subset S$ a monomial ideal. We denote by $G(I)$ the unique minimal monomial system of generators of I. We say that I has linear quotients, if there exists an order $\sigma = u_1, \ldots, u_m$ of $G(I)$ such that the ideal $(u_1, \ldots, u_{i-1}) : u_i$ is generated by a subset of the variables for $i = 2, \ldots, m$. We denote this subset by $q_{u_i,\sigma}(I)$. Any order of the generators for which we have linear quotients will be called an admissible order. Ideals with linear quotients were introduced by Herzog and Takayama [33]. If each component of I has linear quotients, then we say I has componentwise linear quotients.

The concept of linear quotients, similarly as the concept of shellability, is purely combinatorial. However both concepts have strong algebraic implications. Indeed, an ideal with linear quotients has componentwise linear resolutions while shellability of a simplicial complex implies that it is sequentially Cohen-Macaulay. These similarities are not accidental. In fact, let Δ be a simplicial complex and I_Δ its Stanley-Reisner ideal. It is well-known that I_Δ has linear quotients if and only if the Alexander dual of Δ is shellable. Thus at least in the squarefree case "linear quotients" and "shellability" are dual concepts. On the other hand, linear quotients are not only defined for squarefree monomial ideals, and hence this concept is more general than that of shellability.

5.1 Monomial ideals with linear quotients

In this section we prove some fundamental properties of ideals with linear quotients. Let $I \subset S$ be a monomial ideal with linear quotients and $u_1, \ldots, u_m$ an admissible order of $G(I)$. It is easy to see that $\deg u_i \geq \min\{\deg u_1, \ldots, \deg u_{i-1}\}$ for $i = 2, \ldots, m$. In particular, $\deg u_1 = \min\{\deg u_1, \ldots, \deg u_m\}$. But in general, this order need not to be a degree increasing order. For example, the ideal $I = (x_1x_2, x_1x_3^2x_4, x_2x_4)$ has linear quotients in the given order, but $\deg x_1x_3^2x_4 > \deg x_2x_4$.

In the following lemma we show that for any ideal with linear quotients there exists an admissible order $u_1, \ldots, u_m$ of $G(I)$ such that $\deg u_i \leq \deg u_{i+1}$ for $i = 1, \ldots, m-1$. We call such an order a *degree increasing admissible order*.

Lemma 5.1.1. *Let $I \subset S$ be a monomial ideal with linear quotients. Then there is a degree increasing admissible order of $G(I)$.*

Proof. We use induction on m, the number of generators of I, to prove the statement. If $m = 1$, there is nothing to show.

Assume $m > 1$ and $u_1, \ldots, u_m$ is an admissible order. It is clear that $J = (u_1, \ldots, u_{m-1})$ has linear quotients with the given order. By induction hypothesis, we may assume that $\deg u_i \leq \deg u_{i+1}$ for $i = 1, \ldots, m-2$. Assume that $\deg u_{m-1} > \deg u_m$. Let $j+1$ be the smallest integer such that $\deg u_{j+1} > \deg u_m$. By the observation before this lemma, one sees that $j + 1 \neq 1$. Now we show that $u_1, \ldots, u_j, u_m, u_{j+1}, \ldots, u_{m-1}$ is an admissible order which is obviously degree increasing.

We need to prove that $(u_1, \ldots, u_j) : u_m$ and $(u_1, \ldots, u_j, u_m, u_{j+1}, u_{p-1}) : u_p$ are generated in degree one, for $p = j+1, \ldots, m-1$. Since $\deg u_m < \deg u_q$ for $q = j+1, \ldots, m-1$, we have $\deg(u_q : u_m) > 1$. Since $u_1, \ldots, u_m$ is an admissible order, for any $r \leq j$, there exists a $k \leq j$ such that $\deg(u_k : u_m) = 1$ and $u_k : u_m \mid u_r : u_m$. This shows that $(u_1, \ldots, u_j) : u_m$ is generated in degree one. Now let $j+1 \leq p \leq m-1$. It is clear that for any $r \leq p-1$, there exists a $k \leq p-1$ such that $\deg(u_k : u_p) = 1$ and $u_k : u_p \mid u_r : u_p$, since the ideal $(u_1, \ldots, u_j, u_{j+1}, \ldots, u_p)$ has linear quotients in this order. It remains to show that there is an $h < p$ such that $\deg(u_h : u_p) = 1$ and $u_h : u_p \mid u_m : u_p$. Since $u_1, \ldots, u_j, u_{j+1}, \ldots, u_m$ is an admissible order and $\deg u_m < \deg u_q$ for $q = j+1, \ldots, m-1$, there exists a $k \leq j$ such that $u_k : u_m = x_d$ and $x_d \mid u_p : u_m$ for some $d \in [n]$. Since $u_1, \ldots, u_j, u_{j+1}, \ldots, u_p$ is an admissible order, there exists an $h < p$ such that $u_h : u_p = x_b$ and $x_b \mid u_k : u_p$ for some $b \in [n]$.

We claim that $x_b \mid u_m : u_p$. In order to prove this we first show that $b \neq d$. Suppose $b = d$. Then we have $x_d = u_k : u_m$ and $x_d = x_b \mid u_k : u_p$. Hence $\deg_{x_d} u_k = \deg_{x_d} u_m + 1$ and $\deg_{x_d} u_k \geq \deg_{x_d} u_p + 1$, where by $\deg_{x_d} u$ we mean the degree of x_d in u. Therefore $\deg_{x_d} u_m \geq \deg_{x_d} u_p$, which is a contradiction, since $x_d \mid u_p : u_m$.

Now since $x_b = u_h : u_p$ and $x_b \mid u_k : u_p$, we have $\deg_{x_b} u_h = \deg_{x_b} u_p + 1$ and $\deg_{x_b} u_k \geq \deg_{x_b} u_p + 1$. On the other hand, since $x_d = u_k : u_m$ and $b \neq d$, we have $\deg_{x_b} u_m \geq \deg_{x_b} u_k \geq \deg_{x_b} u_p + 1 > \deg_{x_b} u_p$. This implies that $x_b \mid u_m : u_p$. $\square$

If $\sigma = u_1, \ldots, u_m$ is any admissible order of $G(I)$, we denote by $\sigma' = u_{i_1}, \ldots, u_{i_m}$ the degree increasing admissible order derived from σ as give in Lemma 5.1.1. The order σ' is called the degree increasing admissible order induced by σ. With the notation introduced we obtain the following:

Proposition 5.1.2. *Let I be a monomial ideal with linear quotients with respect to the admissible order σ of the generators. Then for all $u \in G(I)$ we have*

$$q_{u,\sigma}(I) = q_{u,\sigma'}(I).$$

Proof. Let $\sigma = u_1, \ldots, u_m$ and $\sigma' = u_{i_1}, \ldots, u_{i_m}$. Suppose $u = u_k$ in σ and $u = u_{i_t}$ in σ'. Let $x_d \in q_{u,\sigma}(I)$, for some $d \in [n]$, then there exists $j < k$ such that $u_j : u_k = x_d$. In particular, $\deg u_j \leq \deg u_k$. According to the definition of σ', u_j comes before u_{i_t} and hence $x_d \in q_{u,\sigma'}(I)$.

Conversely, let $x_d \in q_{u,\sigma'}(I)$ for some $d \in [n]$. Then there exists an i_j with $j < t$, such that $u_{i_j} : u_{i_t} = x_d$. We may assume that j is the smallest integer with this property and $u_{i_j} = u_r$ in σ.

Suppose $x_d \notin q_{u,\sigma}(I)$. Then $r > k$ and $\deg u_r < \deg u_k$ according to the definition of σ'. Therefore $u_r = x_d v$ and $u_k = wv$ where v and w are monomials with $\deg w \geq 2$ and $x_d \nmid w$. Since $u_1, \ldots, u_r$ is an admissible order and $k < r$, there exists an $s < r$

such that $u_s : u_r = x_b$ and $x_b \mid u_k : u_r = w$ $(b \neq d)$. Hence $\deg u_s \leq \deg u_r = \deg u_{i_j}$. Therefore $u_s = u_{i_l}$ with $l < j$.

It follows that $\deg_{x_b} u_s = \deg_{x_b} u_r + 1 \leq \deg_{x_b} u_k$, $\deg_{x_c} u_s \leq \deg_{x_c} u_r \leq \deg_{x_c} u_k$ for any $c \neq d, b$, and $\deg_{x_d} u_s \leq \deg_{x_d} u_r = \deg_{x_d} u_k + 1$. If $\deg_{x_d} u_s < \deg_{x_d} u_k + 1$, then we have $u_s \mid u_k$, a contradiction. Therefore $\deg_{x_d} u_s = \deg_{x_d} u_k + 1$, and hence $x_d = u_s : u_k = u_{i_l} : u_{i_t}$, contradicting the choice of j. □

Let Δ be a simplicial complex with $\mathcal{F}(\Delta) = \{F_1, \ldots, F_m\}$. Then $I_\Delta = \bigcap_{i=1}^m P_{F_i^c}$ where $P_{F_i^c} = (x_j \colon j \notin F_i^c)$ and $F_i^c = [n] \setminus F_i$, see [7, Theorem 5.4.1]. It follows from [28, Lemma 1.2] that $I_{\Delta^\vee} = (x_{F_1^c}, \ldots, x_{F_m^c})$, where $x_{F_i^c} = \prod_{j \notin F_i} x_j$. We follow the notation in [9]: if $\delta = F_1, \ldots, F_m$ is any order of facets of Δ, then we set $\Delta_k = \langle F_1, \ldots, F_k \rangle$ and $R_\delta(F_k) = \{i \in F_k \colon \quad F_k \setminus \{i\} \in \Delta_{k-1}\}$ for any $k \in [m]$.

We observe the following simple but important fact: Δ is shellable with shelling $\delta = F_1, \ldots, F_m$ if and only if $I_{\Delta^\vee}$ has linear quotients with the admissible order $\sigma = x_{F_1^c}, \ldots, x_{F_m^c}$. Moreover, if the equivalent conditions hold, then $R_\delta(F_k) = q_{u_k,\sigma}(I_{\Delta^\vee})$.

As an immediate consequence of Lemma 5.1.1, Proposition 5.1.2 and the observation above we rediscover the following well-known "Rearrangement Lemma" of Björner and Wachs [9, Lemma 2.6].

Corollary 5.1.3. *Let $\delta = F_1, \ldots, F_m$ be a shelling of the simplicial complex Δ. There exists a shelling $\delta' = F_{i_1}, \ldots, F_{i_m}$ of Δ induced by δ such that $\dim F_{i_k} \geq \dim F_{i_{k+1}}$ for $k = 1, \ldots, m-1$. Furthermore we have $R_\delta(F) = R_{\delta'}(F)$ for any facet F of Δ.*

It is known that the product of two ideals with linear quotients need not to have again linear quotients, even if one of them is generated by linear forms. Such an example was given by Conca and Herzog [11].

Example 5.1.4. Let $R = k[a, b, c, d]$, $I = (b, c)$ and $J = (a^2b, abc, bcd, cd^2)$. Then J has linear quotients, and I is generated by a subset of the variables. But the product IJ has no linear quotients (not even a linear resolution).

However, we have the following

Lemma 5.1.5. *Let $I \subset S$ be a monomial ideal. If I has linear quotients, then $\mathfrak{m}I$ has linear quotients, where $\mathfrak{m} = (x_1, \ldots, x_n)$ is the maximal graded ideal of S.*

Proof. We may assume $G(I) = \{u_1, \ldots, u_m\}$ and $u_1, \ldots, u_m$ is a degree increasing admissible order. We prove the assertion by using induction on m.

The case $m = 1$ is trivial. Let $m > 1$. Consider the multi-set

$$T = \{u_1x_1, \ldots, u_1x_n, u_2x_1, \ldots, u_2x_n, \ldots, u_mx_1, \ldots, u_mx_n\}.$$

It is a system of generator of $\mathfrak{m}I$. If $u_ix_j \mid u_rx_s$ for some $i < r$, then we remove u_rx_s from T. In this way, we get the minimal set

$$T' = \bigcup_{i=1}^{m} \bigcup_{j \in A_i} \{u_ix_j\}$$

of monomial generators of $\mathfrak{m}I$, where $A_1 = [n]$ and $A_i \subset [n]$ for $i = 2, \ldots, m$. We shall order $G(\mathfrak{m}I)$ in the following way: $u_k x_l$ comes before $u_t x_s$ if $k < t$ or $k = t$ and $l < s$. Now we show that the above order σ of $G(\mathfrak{m}I)$ is an admissible order. We define the order of the generators of $\mathfrak{m}(u_1 \ldots, u_{m-1})$ in the same way as we did for $\mathfrak{m}I$. Then the ordered sequence τ of the generators of $\mathfrak{m}(u_1 \ldots, u_{m-1})$ is an initial sequence of σ. Moreover, by induction hypothesis, τ is an admissible order of $G(\mathfrak{m}(u_1 \ldots, u_{m-1}))$.

For a given $j \in A_m$ let J be the ideal generated by all monomials in T' which come before $u_m x_j$ with respect to σ. It remains to be shown that $J : u_m x_j$ is generated by monomials of degree 1.

Let $u_k x_l \in G(J)$. If $k = m$, then $u_k x_l : u_m x_j = x_l$. If $k < m$, then we shall find an element $u_r x_s \in G(J)$ and $t \in [n]$ such that $u_r x_s : u_m x_j = x_t$ and $x_t \mid u_k x_l : u_m x_j$. Indeed since $u_1, \ldots, u_m$ is an admissible order of $G(I)$, there exists $q < m$ such that $u_q : u_m = x_t$ and $x_t \mid u_k : u_m$. This implies that $u_q x_j : u_m x_j = u_q : u_m = x_1$. Since $u_q x_j \in \mathfrak{m}I$, there exists, by the definition of σ, a monomial $u_r x_s \in G(J)$ such that $u_r x_s \mid u_q x_j$.

We claim that $u_r x_s : u_m x_j = x_t$ and $x_t \mid u_k x_l : u_m x_j$. Notice that $u_r x_s : u_m x_j \mid u_q x_j : u_m x_j = x_t$. If $u_r x_s : u_m x_j \neq x_t$, then $u_r x_s : u_m x_j = 1$, that is, $u_r x_s \mid u_m x_j$ which contradicts the fact that $j \in A_m$. This shows that $u_r x_s : u_m x_j = x_t$.

Since $x_t \mid u_k : u_m$, it is enough to show that $x_t \neq x_j$ in order to prove that $x_t \mid u_k x_l : u_m x_j$. Assume that $x_t = x_j$. Since $u_q : u_m = x_t$, we have $u_q = x_t u$ for some monomial u such that $u \mid u_m$. Since $\deg u_q \leq \deg u_m$, it follows that $u_m = uw$ for some monomial w with $\deg w \geq 1$ and $x_t \nmid w$. Hence there exists some variable x_d with $d \neq t$ such that $x_d \mid w$. But then $x_d u_q = x_d u x_t \mid w u x_t = u_m x_j$, contradicting again the fact that $j \in A_m$. □

Remark 5.1.6. The converse of the above lemma is not true. For example, let $I = (ab, cd) \subset K[a, b, c, d]$. Then $\mathfrak{m}I = (a^2b, ab^2, abc, abd, acd, bcd, c^2d, cd^2)$ has linear quotients in the given order, but I has no linear quotients.

Now we present the main theorem of this section.

Theorem 5.1.7. *Let $I \subset S$ be a monomial ideal. If I has linear quotients, then I has componentwise linear quotients.*

Proof. By Lemma 5.1.5 and Lemma 5.1.1, we may assume that I is generated by monomials of two different degrees a and $a+1$. We denote by $I_{\langle a \rangle}$ the ideal generated by the a-th graded component of the ideal I. Let $G(I) = \{u_1, \ldots, u_s, v_1, \ldots, v_t\}$, where $\deg u_i = a$ for $i = 1, \ldots, s$ and $\deg v_j = a + 1$ for $j = 1, \ldots, t$. By Lemma 5.1.1, we may assume that $u_1, \ldots, u_s, v_1, \ldots, v_t$ is an admissible order, hence I_a has linear quotients. Now we show that $I_{\langle a+1 \rangle}$ has also linear quotients.

We have $I_{\langle a+1 \rangle} = \mathfrak{m}(u_1, \ldots, u_s) + (v_1, \ldots, v_t)$. Let $G(I_{\langle a+1 \rangle}) = \{w_1, \ldots, w_l, v_1, \ldots, v_t\}$, where $w_1, \ldots, w_l$ is ordered as in Lemma 5.1.5. In particular, $w_1, \ldots, w_l$ is an admissible order. We only need to show that $(w_1, \ldots, w_l, v_1, \ldots, v_{p-1}) : v_p$ is generated by a subset of the variables, for $1 \leq p \leq t$.

First we consider $v_j : v_p$ where $j < p$. Since $u_1, \ldots, u_s, v_1, \ldots, v_t$ is an admissible order of $G(I)$, there exists some $u \in \{u_1, \ldots, u_s, v_1, \ldots, v_t\}$ and $d \in [n]$ such that $u : v_p = x_d$ and $x_d \mid v_j : v_p$. If $u \in \{v_1, \ldots, v_t\}$ we are done. So we may assume $u \in \{u_1, \ldots, u_s\}$. Therefore, $\deg u = \deg v_p - 1$. Since $u : v_p = x_d$, $\deg_{x_d} u = \deg_{x_d} v_p + 1$ and $\deg_{x_b} u \leq \deg_{x_b} v_p$ for any $b \neq d$. Since $\deg u < \deg v_p$, there exists a variable x_c with $c \neq d$ such that $\deg_{x_c} u \leq \deg_{x_c} v_p - 1$. Since $x_c u \in \mathfrak{m}I_{\langle a \rangle}$, one has $x_c u = w_k$ for some $k \leq l$. All this implies that $\deg_{x_d} w_k = \deg_{x_d} u = \deg_{x_d} v_p + 1$ and $\deg_{x_b} w_k \leq \deg_{x_b} v_p$ for any $b \neq d$. Therefore $w_k : v_p = x_d$ and $x_d \mid v_j : v_p$.

It remains to consider $w_j : v_p$. In this case $w_j = x_b u_i$ for some $i \in [s]$ and some $b \in [n]$. Since $u_1, \ldots, u_s, v_1, \ldots, v_t$ is an admissible order, there exists some $u \in \{u_1, \ldots, u_s, v_1, \ldots, v_t\}$ and $d \in [n]$ such that $u : v_p = x_d$ and $x_d \mid u_i : v_p$. Therefore $x_d \mid w_j : v_p$, since $u_i : v_p \mid w_j : v_p$. If $u \in \{v_1, \ldots, v_t\}$, then we are done. So we may assume $u \in \{u_1, \ldots, u_s\}$. Then, as before, there exists a variable x_c with $c \neq d$ such that $x_c u \in \mathfrak{m}I_{\langle a \rangle}$, $\deg_{x_d} x_c u = \deg_{x_d} u = \deg_{x_d} v_p + 1$ and $\deg_{x_b} x_c u \leq \deg_{x_b} v_p$ for any $b \neq d$. This implies that $x_c u : v_p = x_d$ and $x_d \mid w_j : v_p$. $\square$

It is known that if I has linear quotient and generated in one degree, then I has a linear resolution, see [57]. Therefore we get the following:

Corollary 5.1.8. *If $I \subset S$ is a monomial ideal with linear quotients, then I is componentwise linear.*

We do not know if the converse of Theorem 5.1.7 is true in general. However we could prove the following:

Proposition 5.1.9. *Let I be a monomial ideal with componentwise linear quotients. Suppose for each component $I_{\langle a \rangle}$ there exists an admissible order σ_a of $G(I_{\langle a \rangle})$ with the property that the elements of $G(\mathfrak{m}I_{\langle a-1 \rangle})$ form the initial part of σ_a. Then I has linear quotients.*

Proof. We chose the order $\sigma = u_1, \ldots, u_s$ of $G(I)$ such that that $i < j$ if $\deg u_i < \deg u_j$ or $\deg u_i = \deg u_j = a$ and u_i comes before u_j in σ_a.

We show that $(u_1, \ldots, u_{p-1}) : u_p$ is generated by linear forms. If $\deg u_1 = \deg u_p$, then there is nothing to prove.

Now assume that $\deg u_1 < \deg u_p = b$. Let $l < p$ be the largest number such that $\deg u_l < b$. Then, by our assumption, there exists an admissible order $w_1, \ldots, w_t, u_{l+1}, \ldots, u_p$ where $w_1, \ldots, w_t \in G(\mathfrak{m}I_{\langle b-1 \rangle})$.

Let $j < p$ and suppose that $\deg(u_j : u_p) \geq 2$. Let m be a monomial such that $\deg(mu_j) = \deg u_p$ and $mu_j : u_p = u_j : u_p$. Since $mu_j \in \{w_1, \ldots, w_t, u_{l+1}, \ldots, u_{p-1}\}$ there exists $w \in \{w_1, \ldots, w_t, u_{l+1}, \ldots, u_{p-1}\}$ and some $d \in [n]$ such that $w : u_p = x_d$ and $x_d \mid u_j : u_p$ because $mu_j : u_p = u_j : u_p$.

If $w \in \{u_{l+1}, \ldots, u_{p-1}\}$, then we are done. On the other hand, if $w \in \{w_1, \ldots, w_t\}$, then $w = m' u_i$ for some $i \leq l$ and some monomial m'. Since $w : u_p = x_d$, one has $\deg_{x_b} w \leq \deg_{x_b} u_p$ for all $b \neq d$. Hence x_d does not divide m', otherwise $u_i \mid u_p$ which contradicts the fact that $u_i, u_p \in G(I)$. Therefore $x_d = u_i : u_p$ and $x_d \mid u_j : u_p$. $\square$

Let $I \subset S$ be a monomial ideal. We denote by $I_{[sf]}$ the monomial ideal generated by the squarefree monomials in I and call it the squarefree part of I. Indeed $I_{[sf]} = (u\colon u \in G(I)$ and u is squarefree $)$. We follow [21] and denote by $I_{[a]}$ the squarefree part of $I_{\langle a\rangle}$. In [21, Proposition 1.5], the authors proved that if I is squarefree, then $I_{\langle a\rangle}$ has a linear resolution if and only if $I_{[a]}$ has a linear resolution. Indeed for the only if part one does not need the assumption that I is squarefree. We have the following slightly different result.

Proposition 5.1.10. *Let I be a monomial ideal in S. If I has linear quotients, then $I_{[sf]}$ has linear quotients.*

Proof. Let $u_1, \ldots, u_m$ be an admissible order of $G(I)$. Assume $I_{[sf]} = (u_{i_1}, \ldots, u_{i_t})$, where $1 \leq i_1 < i_2 < \cdots < i_t \leq m$. We shall show $u_{i_1}, \ldots, u_{i_t}$ is an admissible order of $G(I_{[sf]})$ by using induction on m.

The case $m = 1$ is trivial. Now assume $m > 1$. It is clear that $(u_{i_1}, \ldots, u_{i_{t-1}})$ is the squarefree part of the monomial ideal $(u_1, \ldots, u_{i_{t-1}})$, where $u_1, \ldots, u_{i_{t-1}}$ is an admissible order. By induction hypothesis $u_{i_1}, \ldots, u_{i_{t-1}}$ is an admissible order of $G((u_{i_1}, \ldots, u_{i_{t-1}}))$. Consider $u_{i_j} : u_{i_t}$ with $j < t$. Since $u_1, \ldots, u_m$ is an admissible order of $G(I)$, there exists $k < i_t$ and some $d \in [n]$ such that $u_k : u_{i_t} = x_d$ and $x_d \mid u_{i_j} : u_{i_t}$. Since u_{i_j} and u_{i_t} are squarefree, we have $x_d \nmid u_{i_t}$. On the other hand, since $u_k : u_{i_t} = x_d$, one has $\deg_{x_d} u_k = 1$ and $\deg_{x_b} u_k \leq \deg_{x_b} u_{i_t} \leq 1$ for any $b \neq d$. Hence $u_k \in G(I_{[sf]})$. $\square$

Combining Proposition 5.1.10 with Theorem 5.1.7, we have the following:

Corollary 5.1.11. *Let $I \subset S$ be a monomial ideal with linear quotients. Then $I_{[a]}$ has linear quotients.*

Remark 5.1.12. Let $E = K\langle e_1, \ldots, e_n\rangle$ be the exterior algebra with basis $e_1, \ldots, e_n$, and $J = (e_{i_1} \wedge \cdots \wedge e_{i_j} \colon 1 \leq i_1 < \ldots < i_j \leq n) \subset E$ a monomial ideal. Suppose $I = (x_{i_1} \cdots x_{i_j} \colon 1 \leq i_1 < \ldots < i_j \leq n)$ is the squarefree monomial ideal in $S = K[x_1, \ldots, x_n]$ corresponding to J. It is easy to see that J has linear quotients if and only if I has linear quotients. Hence as an immediately consequence of Corollary 5.1.11, one sees that if J has linear quotients, then each component of J has linear quotients.

Let Δ be a d-dimensional simplicial complex. We define the 1-*facet skeleton* of Δ to be the simplicial complex

$$\Delta^{[1]} = \langle G\colon G \subset F \in \mathcal{F}(\Delta) \text{ and } |G| = |F| - 1\rangle.$$

Recursively, the i-*facet skeleton* is defined to be the 1-facet skeleton of $\Delta^{[i-1]}$, for $i = 1, \ldots, d$. For example if $\Delta = \langle\{1,2,3\},\{2,3,4\},\{4,5\}\rangle$, then

$$\Delta^{[1]} = \langle\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{3,4\},\{5\}\rangle \text{ and } \Delta^{[2]} = \langle\{1\},\{2\},\{3\},\{4\}\rangle.$$

If Δ is pure of dimension d, then the i-facet skeleton of Δ is just the $(d-i)$-skeleton of Δ. Now let Γ be a shellable simplicial complex with facets $F_1 \ldots, F_m$. It is known that any skeleton of Γ is shellable, see [9, Theorem 2.9]. Since $I_\Gamma = \bigcap_{i=1}^m P_{F_i}$ where $P_{F_i} = (x_j \colon j \notin F_i)$, we have $(I_\Gamma)^\vee = (u_1, \ldots, u_m)$, where $u_i = \prod_{j \notin F_i} x_j$. By Theorem 1.4.6 $(I_\Gamma)^\vee$ has linear quotients. Hence $\mathfrak{m}(I_\Gamma)^\vee$ and the squarefree part of $\mathfrak{m}(I_\Gamma)^\vee$ have linear quotients by Lemma 5.1.5 and Proposition 5.1.10. It is not hard to see that the squarefree part of $\mathfrak{m}(I_\Gamma)^\vee$ is the Alexsander dual of $I_{\Gamma^{[1]}}$. hence with the discussion above, we get the following:

Corollary 5.1.13. *If Γ is a shellable simplicial complex of dimension d, then $\Gamma^{[i]}$ is shellable, for $i \leq d$. In particular, if Γ is pure, then any skeleton of Γ is again shellable.*

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